Rational Counting—Seat Written Work.

Children putting into written number language the measurements which they have expressed on their desks in objective language (Section 13 (*b*) and (*c*)).

The three columns of concrete work indicate the form in which the teacher arranged the "blackboard dictation" for this particular exercise. The dictations, the objective work, and the seat written work should be alike in column arrangement in all cases (Section 12 (*d*), note).

The constructions are on the left side of the desk, so as to leave the right side clear for use in the written work (Section 7 (*b*), note).

The teacher's duties in connection with the work are—

1. To prepare the blackboard dictation.
2. To inspect the concrete work.
3. To inspect the written work (Section 8 (*c*), Note 1).

NUMBER BY DEVELOPMENT

A METHOD
OF NUMBER INSTRUCTION

Primary

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APPROXIMATE TIME SCHEDULES.

As a further guide to the work in this Outline the following time schedules are suggested for the principal steps. It must be understood that no one can fix the time for any part of this work that could be followed by all teachers. Classes differ in ability; teachers of equal power differ in the time that they demand for the development of a given subject or element. The class and the teacher are, therefore, factors in this matter of rate of progress. A time schedule for this reason can be merely suggestive and approximate.

As the work assigned to each grade can easily be done in a school year of 36 weeks, the schedule is based upon that length of term.

For Grade One.

Sections 1–7 will require 7 weeks.
Section 8 may be begun about the 8th week.
Section 9 may be begun about the 10th week.
Section 12 may be begun about the 12th week.
Section 27 may be begun about the 15th week.
Section 28 may be begun about the 16th week.
Section 29 may be begun about the 17th week.
Section 30 may be begun about the 20th week.
Section 35 may be begun about the 24th week.
Section 50 may be begun about the 28th week.
Section 57 may be begun about the 30th week.
Section 60 may be begun about the 32d week.

For Grade Two.

Sections 64–69 will require about 26 weeks.
Sections 77–101 will require about 10 weeks.

For Grade Three.

Proper-fraction partition should be given about 20 weeks.

"Reversed" and general work should be given about 4 weeks.

Improper-fraction work should be given about 8 weeks.

Oral and written mixed partition should be given about 4 weeks.

INTRODUCTION

THE aim of this book is to present a working outline of a development system of primary number teaching. Development work in any branch of instruction depends for its value upon the care that is given to analysis, arrangement, and presentation. The separation of the subject into elementary steps, the careful consideration of each step to make sure that it is an element and not a compound, the pedagogical arrangement of these elements, and their presentation in this order with proper method of approach to each, are the factors in a development process—hence manifestly matters of extreme detail. Give the teacher merely the outlines of a system of instruction by development in any subject, even one with which she is perfectly familiar, and require her to work out the details for herself, the difficult, the delicate part of the task is thrown upon her. A mistake in this analysis or in the arrangement of the elements, and the work ceases to be a development process. System ceases. It is on this account that the various teaching steps, which

have required years of experiment and study to determine and arrange, are given so fully.

The General Plan.

The general teaching plan of the system may be seen in the three steps in the development process of each department of the work:

(*a*) To teach the child the oral and written language of the process or processes under development.

(*b*) To give him experience in the use of this language through seat work in translating written into concrete expressions and his concrete work into written form.

(*c*) To give him experience in the use of this language applied to seat work counting, the child through this experience gradually acquiring a knowledge of the "number facts" without the drills and coaching which such knowledge usually requires.

The Form of the Work.

The method is purely objective—planned with a view to give the pupil "occupation" counting, which he may do without the constant oversight of the teacher. The aim is to give the minimum of teaching with the maximum of independent work by the pupil.

The child gains his number knowledge, not from drills and recitations, but through personal experience in number construction work —counting with group units—at his own desk, under the impulse of interest in the work, during periods which might otherwise be wasted or given to seat "occupations" of doubtful value. No subject in the school curriculum lends itself more perfectly to this laboratory form of work than number.

It has been necessary, in the development of this system of number training, to keep in view the following educational demands, to which any plan of number teaching to be worthy of attention must conform:

1. That it must make use of a natural activity with natural material.

2. That all concrete work must be true to the measuring processes.

3. That the objective work must be of such form and nature as to give the child true number concepts:

(*a*) It must develop that 3, or 4, or 6 is a group, and that it is a unit, *a* 3, or *a* 4, or *a* 6. The work must produce a habit of putting such expressions into unity form as a means to the final recognition of their group-unit nature.

(*b*) It must lead to the recognition of the principle that in any given expression the measuring units must be equal. To this end the construction work must require an act of choice in this respect —the selection of similar and equal parts for each concrete expression.

4. That from the beginning the language used, however changed in form to meet the needs of the primary child, shall be mathematically accurate.

5. That the pupil's constructions or measuring work throughout the whole development period must be of such a nature and form as to leave no doubt that the mental measuring act which accompanies each act of construction is that which the step is designed to give, and that this mental product will be the same whether the child is working alone or under the eye of the teacher. This is the test by which object work—laboratory work—in number is distinguished from that which is merely objective illustration.

The author desires to express his thanks to the scores of practical, progressive teachers who have aided him by experiment and suggestion in the development of this number system, but particularly—

To Mrs. Edith G. Johnson-Malott, of Abilene, Kansas, for much of the primary language, and for practical school-room work which first suggested the possibility of a development system in number.

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J. C. G.

NUMBER BY DEVELOPMENT

PART I

PRINCIPLES APPLICABLE IN NUMBER TEACHING

I. Origin of the Measuring Instinct.

(*a*) The power to measure and the instinct to claim ownership in toys have a common origin. They grow out of limitations in the number of the child's playthings. The instinct to own is developed when these objects, whether many or few, are dissimilar (the child having but one toy of each kind), or when the child has a small number of similar playthings—toys so much alike as to be indistinguishable by the senses. It is in this latter case that measuring power is also developed.

1. In the case where the toys are dissimilar the instinct of ownership is developed but not the power to count. No matter how many such toys he may have, the child recognizes

and becomes attached to certain individuals which please him and thinks of each one, "This is mine." If one of them is taken without his knowledge and he detects the loss, the thought is of the loss of a particular individual. This recognition of the individual playthings makes counting unnecessary; hence his play activities under such conditions are without stimuli for the development of measuring power.

2. On the other hand, when the toys are without distinguishing qualities the detection of the loss of a toy must be through a quantitative idea. For illustration: the child is playing with two red balls similar in all respects. Some one diverts his attention for a moment while we take one away and secrete it. If, on turning again to his play, the child detects the loss, it must be by an act of measuring. Having no means of distinguishing the one from the other, his thought could not be "I have lost this" or "I have lost that." Let his attention be again diverted and quietly replace the ball. If on turning to his play he is satisfied and ceases his search for lost property, we must conclude that the mental act by which he detected the loss was not only quantitative but definitely so.

(*b*) But it is only when the number of toys is limited that the power to measure develops. The child that has none and the child that has more than his needs demand are without stimuli to arouse the measuring instincts. There is a period in the child's life when he can make use of but one toy. Give such a child a second toy and the first is abandoned, at least for a time. Then comes the two-toy, then the three-toy age.

(*c*) It is in this two-toy period that for the first time conditions may exist which will bring into activity the measuring tendency. If the child has two toys, his play demands require both. If he loses one of them, his play tendencies are without the material for their full exercise, there is an unsatisfied want, he detects the loss. There is no lack of quantitative definiteness in this loss, because if the toy is restored the child is satisfied. In this detection of loss is the birth of a new power—the power to measure. It came into activity because the toys were limited to his bare needs. Give him a larger number—a dozen only, perhaps—and his possessions are so far beyond his needs that in his mind the supply is unlimited, the number infinite. Quietly take one of them away, then another, then

another—there will be no sense of loss, his plays will be unhampered, the tendency to own not excited, the measuring power not brought into exercise, until this subtracting process brings the number of toys again within the limit of his play needs, when loss will be detected—measuring power will become active. It is under these conditions of limit and similarity that the tendency to count is aroused.

(*d*) This measuring is at first in terms of less—"I have less than I had beforc." The idea is quantitatively definite, but the quantitative terms one, two, three, etc., are wanting. Numerical language is acquired later, just as spoken words of any kind are a later development than the corresponding ideas.

(*e*) Measuring is, therefore, one of the early intellectual activities. It is nascent at an age long before the primary school period, perhaps before the period of spoken language.

(*f*) This is suggestive of what the primary school should do in the matter of this line of training. There is a principle which is generally accepted as fundamental, that the training of a faculty or tendency should begin when it first becomes active and that if this is not done there will be a tendency to atrophy.

(*g*) This principle should be taken seriously, it should have practical recognition, in school administration. The number instinct is active when the child first enters school, and to neglect or defer number training limits the possibilities of development in a most important form of activity.

II. The First Quantitative Ideas.

(*a*) The child's first measuring act, as we have shown, is with *two*. Later he reaches the three-toy period and his measuring power extends to *three*. It is very doubtful if the general idea *one*—a single thing considered quantitatively—has a place in the child's mind before the very much later period of spoken language. The child may have two apples. If he loses one, his detection of the loss, as has been shown, is due to the fact that he knows when he has two and when he has less than two. But he thinks of his loss as *an* apple, not *one* apple. He has no need for a unit of measure, a *one*, and no such need will be felt until after he reaches the three-toy period and recognizes three and less than three. The order of development, therefore, of measuring ideas is two, three, one.

(*b*) This is a very important consideration

in introductory work. Number work of any kind with two and three is more elementary and demands less maturity than similar work which makes use also of *one*. An apple, a book, or a ball is not in the mind as *one* until we think of it as a unit of measure—a unit by which to measure a quantity of apples, books or balls. One thing placed before an individual old or young does not provoke a quantitative idea unless How many? is directly or indirectly suggested. A marble is *this* or *my* marble, not *one* marble. "The book, the crayon, or the cube is a unity, a whole, in itself, but it is not a unit save as used to count up (value) the total amount." (Psychology of Number, by McLellan and Dewey, p. 81.)

(*c*) *One* should be approached with caution in early number training, because of this absence of quantitative suggestion in a single object before the senses. Two twos, two threes, and three twos are very simple conceptions at the primary school age if the child is led up to them in a proper manner; expressions like three ones or two ones are more difficult; very much more difficult still is one three or one two or one one.

(*d*) Number work, therefore, should begin not with one, but with two, three, and four, and should deal with twos, threes, etc., not

with *ones*. A single group to be measured alone—one two, one three, etc.—is not a necessary development, and should not be introduced until after the child has had several weeks of language and counting experience, and even then with caution.

III. Kinds of Measuring Units.

(*a*) Measuring units are classified as to the definiteness with which they measure quantity as quantitative or qualitative.

(*b*) A quantitative unit is one that expresses a definite quantity—quart, bushel, pound, linear foot, etc.

(*c*) A qualitative unit is one individual of a class of objects used as a unit in measuring a quantity of its own kind—egg, chair, orange, spool of thread, sheet of paper, etc. A dozen eggs, three oranges, a ream of paper, a gross of pens are definite as to *how many* eggs, oranges, sheets of paper, etc., but indefinite as to volume or weight.

(*d*) In working out a method of number teaching that will follow "the natural grain of the mental structure" and contribute to the "straightforward workings of the mental machinery," one must make sure that every factor in his completed plan conspires to make

the method a form of natural activity working upon natural material. He must not assume that it is only necessary that the child measure, the form of this measuring work being immaterial, or that the use of either quantitative or qualitative measuring units will satisfy all conditions, there being no reason in pedagogy for the use of the one that does not apply with equal force to the use of the other. The question must be asked and settled as to whether there are any grounds for choice, and, if so, which is the better for use in the proper development of number language and imagery. While teachers differ in this matter—some basing their work wholly upon one, others upon the other—it is far from being an open question and should be given most careful consideration.

(*e*) There are several essential requirements, some relating to number, others to teaching, which must be provided for in every system of introductory work. Three of these must be considered in deciding this question of the measuring units:

1. The teaching requirement—that there shall be nothing in the instruction given or in the imagery or language developed that the child must later unlearn or that is not math-

ematically true. The language must be in accord with the imagery and both must be strictly mathematical.

2. The mathematical—that the measuring units shall be equal. In the expression 4 pounds the measuring units are pounds, in 7 quarts they are quarts. The mathematical requirement is that in every such expression (in every expression of measured quantity) the measuring units shall have equal quantitative value—the pounds must be equal and the quarts equal.

3. The economic—that the measuring units shall express definite quantities—quantities whose values are understood more or less widely.

(*f*) The first and third of these requirements are absolute. The second is somewhat modified in commercial usage, but will not admit of modification and can not be disregarded during the number language period. Eggs are bought and sold with egg as the measuring unit, notwithstanding the fact that egg does not represent a definitely measured quantity in either volume or weight. The same is true of scores of other articles in common use—oranges, lemons, balls, chairs, marbles, etc. The individuals of each of these classes vary in volume and weight, but com-

mercial usage *considers* them equal and thus makes them units of measure.

(*g*) Now, the power to understand what is meant by "considering" a class of non-uniform objects equal is the result of mathematical training and experience. It involves a sense of the need of equal measuring parts—the feeling of impropriety in having the parts unequal. This mental state is the result of number language training in which the child is forced to consider the equality of the parts in the construction of every concrete expression. If we should say to a child who through his whole number language development period has used a mixed assortment of objects and has been required to construct his concrete expressions (4, 3×5, $8-3$, $14 \div 5$, $\frac{3}{4}$ of 12, $11 =$, etc.) during this period so that each shall have similar and equal objects through all its parts, being compelled to select from his supply of mixed units those in each case that will fulfil this requirement, and has done such construction work until his sense of propriety would be shocked by unequal objects in any given expression—if we should say to such a child, when we are about to construct an expression with *unequal* objects (with books, apples, or mixed marbles), "Let us

play that these objects are equal," he would understand fully *why* such a suggestion is made and the need of making it when using such objects, and would intelligently "play" or "consider" as directed.

(*h*) On the other hand, the child who has not had number construction experience, or has used but one class of objects, or has been permitted to construct expressions without regard to similarity and equality of parts, has not had the experience to give him the mental habit necessary to "consider" or "play" equality of parts, no training that could afford a basis for even comprehending the term. The child "sees" in a construction only those features of the work in which he is compelled by the conditions of the construction to exercise *choice*. If, therefore, he has been using but one class of objects (unmixed objects), although his concrete expressions have been formed of equal parts, he has had no necessity for *choosing* like objects, and the equality of the parts has not been in the thought product of the work nor been observed.

Any plan of work which fully meets the second "requirement" would as fully meet the third. This will be considered in the discussion of qualitative units. On the other

hand, a plan which fully provides for meeting the third "requirement" might have in it nothing that would give the child a hint of the principle involved in the second.

IV. Quantitative Units for Development Work.

CLASSES OF UNITS.

(*a*) Of units of this kind four only are available for development uses—the linear inch with inch-sticks, the surface inch with tablets, the cubic-inch with inch blocks, and weight units—ounces, grains, scruples, pennyweights. It needs but a hasty consideration of the character or volume of the various other quantitative units of measure in common use—units of value, liquid units, dry units, etc.—to see that none of them are practicable for objective work in the hands of the pupil.

Linear Units.

(*b*) In the use of the linear inch as an *inch* it would be almost impossible to avoid the development of false mathematical concepts. The child's nature leads him to count *things*. His plays, playgrounds, or playthings have not required the laying out or measuring of definite lengths. He has had no occasion to ask or think How long? or How wide? The

counting which his social life has required has been wholly in the how many of things or persons. He gets his first linear experiences in school. When, therefore, he begins to handle quantities of inch-measures every tendency of nature and experience leads his thoughts to the *objects* rather than to their *lengths*.

Objective work in which the principle thought element desired is thus external to the construction is of very doubtful value. To think the *stick* itself when the language reference and the aim of the construction are to its *length* is bad, either as English or as mathematics, and results in something that must later be unlearned. To ensure the proper thought product the teacher must be constantly at the pupil's side in his construction work to guide him away from the *thing* to its *length*; and, even after long-continued and careful personal attention of this kind, she can have no assurance that the pupil's mind, whenever he is left to himself in seat counting work, will not revert to its natural tendency to count *things*.

Weight Units.

(*c*) The use of ounce, grain, pennyweight, or scruple weights would be open to similar objections. The child's thought would be

upon the *object* almost unavoidably, while the language would apply to *weight.*

Surface Units.

(*d*) Of all the standard measuring units seemingly available for primary development work the square-inch is most productive of false mathematical concepts and looseness in language. For the child there is something peculiarly obscure in *surface.* It is common experience with teachers that it is a matter of no little difficulty to teach older pupils—those in intermediate grades—to differentiate clearly the surface from the object itself. How much greater, almost impossible, the problem with pupils of the primary age! The child is required to think *surface.* He is given language that applies to surface. The square-inch tablet whose surface we would have him "see" and regard as his unit, his one, has *two* unit surfaces, not including the surfaces of its narrow edges. If the child really acquires the true concept of surface and surface inch, it is not easy to understand how he can avoid the thought "on this side" or "on one side" when he uses the term square inch in connection with his object. The teacher who believes that her primary pupil is thinking and count-

ing square inches and not *tablets* is deceived. The child is too young to break away from his natural instinct to count *things*.

Cubic-inch Units.

(*e*) It is obvious that in the use of the cubic inch there would be a similar tendency to concentrate the attention upon the *objects* and count *them*—to think *thing* rather than *volume*. The objection to this is not so much because the concept would be imperfect as on account of its narrowness. It would not be a volume, but a *thing* having a certain shape. The child would not recognize that the term cubic-inch which he has applied to this block would be applicable also to the capacity of a cubic-inch cup or to an equal volume of matter not in cubical form. The gain from the use of this measuring unit, therefore, in the way of becoming familiar with it practically, would be very slight, because it would still be necessary, on account of the extreme narrowness of the concept, to develop its real significance at a later period of the child's school life. Considering this unit in as favorable light as possible, even overlooking the narrow and imperfect imagery that it would develop, there is still the objection to its use which it

has in common with all other quantitative units, that with it we can not meet the second or mathematical "requirement."

Variety of Units Impossible.

(*f*) To meet the second "requirement" there must be, as has been shown, a variety of units and constant necessity for choice in the construction work. Quantitative units, however, must be used alone. The child of the primary age can not use two or more of them together. To teach a child to differentiate quantity (length, surface, weight, or volume) from the object is a matter of no small difficulty. To teach him two of these quantitative units is doubly difficult. More difficult still is the task of developing in him the power to use two of these classes of units together—constructing now with reference to volume and now with reference perhaps to length. To develop and fix the quantitative in the cubic inch, for illustration, it must be made the sole unit, so that all thought in the construction work shall be of *volume*. There could be no hope of establishing the habit of thinking volume during this early development period if parallel work were attempted in which the child must think length or weight.

Conclusions.

(*g*) The use of standard measuring units for number development work is, therefore, indefensible, for the following reasons:

1. It is contrary to nature, in that it is an attempt to have the child consider and count abstract and obscure qualities apart from the objects.

2. It develops false or imperfect mathematical imagery and language.

3. With such units it is impossible for the child to learn through construction experiences to think of the measuring parts of an expression as being necessarily equal.

V. Qualitative Units.

(*a*) There are several considerations which commend these units for development work—

1. Measuring with units of this kind consists in counting *things*. It is a natural activity on natural material—the form which educational demands prescribe for true objective work.

2. They are the units by which the race developed mathematical language and concepts. Nearly all the standard units of weight, length, volume, etc., are former qualitative units which industrial and commercial

needs forced nations to standardize. Quart, foot, pound, meter, are illustrations of such development processes. The foot was the length of the king's foot in each kingdom, the yard King Henry's reach from nose to thumb, the meter the length of a certain bar of platinum (theoretically quantitative, practically qualitative) taken as a standard.

3. Their use does not lead to imperfect imagery. There is no uncertainty in the mental product of a natural activity.

4. With them there is no obstacle to the use of mixed objects, thus providing for the fulfilment of the second "requirement."

(*b*) The possibility of their meeting the third "requirement" is the only question that remains for consideration. The only advantage in the use of quantitative units, even were their use practicable, would be in familiarizing the pupil with them as quantities in common use. There is no special virtue in them excepting this definiteness in the commercial world. During this development period the child's mathematical world is the school-room, the interested parties the other pupils in the room. A unit that is definite there answers for the time every mathematical or economic demand. Lentils made by ma-

chinery, perfect and uniform as such machine objects usually are (all imperfect lentils if there be such being carefully discarded by the pupil whenever found), fully meet the requirements of quantitative definiteness. The quantitative value of "one lentil" is understood within the school-room community or the school community where such objects are in common use. In fact, "lentil," if we select a special kind, could be standardized as a unit of measure with as much propriety as the length of a certain king's foot.

(*c*) Our only concern now is for units for development purposes. Once developed, language and imagery are the same whatever the units: $2+3$, 4 5's, or $\frac{3}{4}$ of 8 could be made concrete with quart units, pound units, foot units, or shoe-pegs. The imagery when this expression is applied to pegs differs from that when it is applied to pounds of iron only in the character of the unit. There is nothing excepting the unit itself in the imagery of a peg, a lentil, or an inch-stick unit that is not used when the expression is applied to feet or pounds. The child, therefore, who has developed his language and imagery with these qualitative units, at a later period applies this language to expressions in quantitative units without

even recognizing that he is in a new field of applied language. He has had a variety of units during this development period, and a new one, if he understands what it means, is brought into use with hardly a thought that it is strange.

(*d*) It is in the use of qualitative units only, therefore, that we may make use of natural activities with natural counting material, and at the same time meet every demand for accurate mathematical language and clear and true mathematical imagery.

VI. Number as Group-units.

(*a*) There is no more important matter in introductory number work than the development of the real significance of the number terms—two, three, four, five, etc. The natural counting before the period of systematic and proper number training recognizes but one kind of unity—the single thing. The child's two is two *things*, his three three *things*. The province of the school is to lift this counting to a higher plane, to develop a form of counting that will give number its proper meaning. The imagery which school work develops from the very first should be true to science.

(*b*) To give the child the true number concept two must mean a unity, a whole, made up of two measuring parts; three must mean in the same way a three-group; four a four-group; etc. "The concept two involves the act of putting together and holding together the two discriminated ones." (Psychology of Number, p. 31.) "In the simple recognition, for example, of three things as three the following intellectual operations are involved: *The recognition of the three objects as forming one connected whole or group*—that is, there must be the recognition of the three things as individuals, and of the *one*, the unity, the whole, made up of the three things." (Psychology of Number, p. 24.)

(*c*) The recognition of this group-unity nature of 2, 3, 4, 5, etc., is therefore essential to a true number concept, and the method of training which fails to give the child this view of number leaves out a vital element of the number imagery. The school number work must bring the child to regard an expression like 3 + 4 as a three-group and a four-group, 2 5's as an expression of two equal five-groups.

(*d*) The development of the recognition of these numbers as expressing group units is not a single teaching step—an element—in

the sense that it must have a separate presentation, or a special place in the arrangement of teaching steps. Neither can it be explained to the child; he has had no experiences with which to interpret group-unit explanations. No essential element of number imagery can be given to the child except through his constructive experiences.

It is a teaching axiom that whatever is essential to the concept is essential to the percepts from which it develops. This axiom is in another form, that a child sees in a construction of any kind only what he consciously puts into it, each act of the construction giving rise to one of the elements of the thought product of the work.

To apply this to this matter of the development of the group-unit nature of number: if the percept—the concrete expression—is to be thought of as a group, the child must be required (it may be an unspoken requirement), as a necessary part of the act of making it, actually to put the constituent units of the expression into group or bundle form. Construction work must be closely watched in this respect, and where the units are not thus in contact the child's attention must be called to the fact by the suggestion, "You must

make them touch." The reason for this can not and need not be given him, but the effect of this unvarying requirement in his construction work will be to make his number imagery take on the group form—the form which his concrete work, the basis of his imagery, has invariably taken.

It must be apparent, therefore, that for this development, just as for any other essential element of the number imagery, the character of the objective work and the form of the objects must be given most careful consideration, in order that we may be assured that the work gives the child the percept desired.

VII. The Objective Work in Group-unity Development.

(*a*) Every form of constructive activity has its natural, accompanying, mental activity—any change in the form of the work changing this thought product. There may be one or more than one thought element in this product, according to the simplicity or complexity of the construction.

The line of thought which thus naturally accompanies a given kind of constructive activity is said to be *internal* to the work. To attempt to use a given form of objective work

for any other thought purpose than that which is internal to it would require the constant presence of the instructor to put the desired thought into each construction as the child makes it. A line of thought which is thus put into the construction artificially is said to be *external* to the work.

(*b*) The difference between an *internal* and an *external* thought product is one of certainty. When the psychical product desired is internal, the thought purposes of the construction are realized if the pupil does the work. "Laboratory work," so called, on any subject is of this kind—the work yielding a thought product whose nature is known and which is an inseparable accompaniment of the work. Constructive work which yields an *external* mental product is illustrative rather than objective in the full sense of the term, and the character or clearness of the child's mental picture in any case is something of which the teacher can have no certainty. What the teacher attempts to put into the child's mind in connection with the work with the objects is one thing, the child's real thought processes may be a very different thing in so far as they relate to the essential elements of the development.

(*c*) A plan of number teaching, therefore, may entirely miss its purpose through the use of objective work to which the desired thought product is not internal.

VIII. Form of Objects for Group-unity Development.

(*a*) We have shown in another connection that standard measuring units fail to meet requirements for developing some of the essential elements of number imagery. It remains to consider objects (sticks, tablets, or blocks) which represent more than one unit—two-inch sticks or two-inch blocks to represent two, three-inch sticks or blocks to represent three, etc.—with reference to their value for the development of this group-unit nature of number.

If we construct 2 3's with 1-inch sticks, or with any other form of single objects, the whole concept is represented in the percept—the 2 is represented by two groups, the groups are 3's in fact. If we attempt to construct the expression with two 3-inch sticks or tablets or blocks, the "2" part of the concept is in the percept but the idea "3's" is external to the construction. The object which we would call *a three* is *one* in the child's experience, *one* in the minds of ordinary people, *one* mathe-

matically. If the child or the man thinks of it as *three*, this three-notion must be put into it for reasons which do not appear in the object itself. Our child of six or seven is in the perception, not the reasoning, period of life. He may put the thought (the word) *three* into this object because the teacher at his elbow insists upon it, but this stick—this physical unit—does not have the qualities to make the child think *three*. "There must be enough qualitative unlikeness—if only of position in space or sequence in time—to mark off the individual objects, to keep them from fusing or running into one vague whole. Part of the difficulty of performing the abstraction which is required to get the idea of number is, accordingly, that this abstraction is complex, involving two factors: the difference which makes the individuality of each object must be noted, and yet the different individuals must be grasped as one whole—a sum." (Psychology of Number, p. 25.) "If two things are simply *fused* in each other, forming a sort of *oneness*, or if they are simply *kept apart* from each other, there is no counting, no 'two'." (Psychology of Number, p. 31.)

Numerical ideas, true counting, can not be developed with these physical wholes repre-

senting 2, 3, 4, etc. "This operation involves (*a*) *discrimination*, or the recognition of the objects as distinct individuals (units): . . . and (2) *grouping*, the gathering together of the like objects (units) into a whole or class, the *sum*." (Psychology of Number, p. 32.) "In this process there must be sufficient qualitative difference among the objects used to facilitate the recognition of individuals as distinct, but not enough to resist the power of grouping all the individuals, of grasping them as parts of one whole or sum." (Psychology of Number, p. 32.)

(*b*) Physical wholes, therefore, single objects used to represent more than a single number, can not be used for group-unit development. The percepts are wanting in the group character which they should give to the concept. This leaves us for use, in this introductory stage of number training, simply *things*—pegs, inch sticks, lentils, etc. It is with such objects only that we can construct number expressions in which the thought products essential to the true number concept are internal.

PART II

OUTLINE OF THE STEPS IN THE DEVELOPMENT PROCESS

The outline here given contains the separate steps that must be taken, arranged in their proper sequence, to ensure a pure development process. The work on what is known as the "number facts" is divided into parts as follows:

1. The facts in addition, subtraction, multiplication, and division, of units of the first order.

2. The facts in the same processes, of units of the second order.

3. The facts of 10–20.

4. The facts of partition.

The plan followed in the development of each of the five processes (addition, subtraction, multiplication, division, and partition) is as follows:

1. The oral and written language.

2. Seat work in the use of this language.

3. Counting in which the child uses the language.

CHAPTER I

THE LANGUAGE OF + AND 'S

SUMMARY:

Reading and writing the numerals—Section 1.

The development of the oral and written language of rational (or group) counting, bringing into use expressions in "and" (+) and 's—Sections 2 and 3.

Reading concrete expressions in + and 's—Sections 4 and 5.

The construction of concrete expressions in + and 's—from oral dictations (Section 6), from written dictations (Section 7.)

1. Reading and Writing Numerals.

Teach to read and write the numerals 1, 2, 3, etc., to 10; or, if the child can not count to 10, to his counting limit. This work should run side by side with that in Sections 2 and 3, of which it is pedagogically a part.

NOTE.—There is nothing in the nature of the work of this outline to Chapter IV that requires a counting range beyond 1–4 or 1–5. If the child's range is found to be only 1–4, the work outlined in Sections 2, 3, 4,

etc., should be confined to these numbers; but as soon as his counting limit has advanced to 5 and 6, etc., these numbers one after the other should be included in the daily work until all the numerals from 1 to 10 are in use. It will be necessary, therefore, to test the pupil frequently as to his power to count, in order that the working range may keep pace with the advance of the counting limit and cover the full range 1–10 as soon as possible. Ordinarily no work need be done to extend the counting range. This will take care of itself through the associations of school life. The teacher's responsibility lies in detecting and making use of each step in the child's advance in this power to count.

2. Introduction to the Oral and Written Language of Rational Counting.

In Section 1 the child learns to read and write *words;* in this section, *phrases.* This written language will require the teaching of the sign + (and) and the words formed with 's in connection with the numerals—4's, 3's, 5's, etc.

Teaching Plan.

The teacher takes a group of pupils to the work-table, the group to be no larger than can stand at the table and work freely. There should be a plentiful supply of large regular objects on the table, such as inch cubes, 3-inch or 4-inch kindergarten sticks, or splints from

4 to 6 inches long. The exercises following will illustrate the method of development—

(*a*) "Show me two blocks" (or splints). 

"Show me another two." 

"Show me another two." 

"Now, how many twos have you?" ("Three twos.")

"Count them." (Here the child points to each of his twos in order, beginning of course at the left—"one, two, three—three twos.")

"This is the way the crayon says three twos"—(3 2's) (written by the teacher on the blackboard).

NOTE.—It is understood, of course, that after each of these constructions, (*a*), (*b*), (*c*), etc., and before the next is dictated, each pupil must clear away his objects ready for the new constructions.

(*b*) "Show me three." 

"Show me another three." 

"Now, how many threes have you?" ("Two threes.")

"Count them." ("One, two—two threes.")

"This is the way the crayon says it"—(2 3's).

(*c*) "Show me a four." ()

"Show me another four." ()

"Show me another four." ()

"Now, how many fours have you?" ("Three fours.")

"Count them."

"This is the way the crayon says it"—(3 4's).

NOTE.—As soon as practicable the teacher, instead of telling "how the crayon says it," asks the pupil to show on the blackboard how the crayon says it.

(*d*) "Show me a four." ()

"Show me a two." ()

"Now, what have you?" ("A four and a two.")

"This is the way the crayon says it"—(4 + 2).

(*e*) "Show me a two." ()

"Show me a five." ()

"Now what have you?" ("A two and a five.")

"This is the way the crayon says it"—(2 + 5).

(*f*) "Show me a five." ()

"Show me another five." ()

"Now, what have you?" ("A five and a five.")

"Make the crayon say it." (5 + 5.)

"How many fives?" ("Two fives.")

"Count them."

"Make the crayon say it in that way." (2 5's.)

NOTE.—It is very important that a beginning be made of reading in two ways expressions made up of equal groups. The objective expression for three twos

()

should be seen as three twos or again as a two and a two and a two. There should be a way of speaking of each of these readings. We might call "3 2's" the short way and "2 + 2 + 2" the long way; or, the former might be called the "'s" way, the latter the "and" way—a "'s" story and an "and" story.

3. The Dictations Given as Wholes.

Very soon the dictations may be given in full at one time.

(*a*) "Show me a three and a two." ()

"This is the way the crayon says it," or "Show me how the crayon says it." (3 + 2.)

(*b*) "Show me three fours." ()

"What have you?" ("Three fours.")

"Count them."

"Show me how the crayon says it."

"Read it 'the long way'" (or "as an '*and* story'"). ("A four and a four and a four.")

"Say it the long way (or "as an *and* story") with the crayon." (4 + 4 + 4.)

(*c*) "Show me a three and a three and a three." ()

"Show me how the crayon says it."

"Read it as a 's story" (or "the short way"). ("Three three's.")

"Make the crayon say it this way." (3 3's.)

(*d*) "Show me a two and a two and a two." ()

"Make the crayon say it." (2 + 2 + 2.)

"Read it the short way." ("Three twos.")

"Make the crayon say it the short way." (3 2's.)

4. Objective Work by the Teacher for Pupil to Read.

This is the converse of Section 2.

This section contemplates—

(*a*) The oral reading or interpretation of objective work.

1. Reading *all* expressions "the long way" as "*and* stories."
2. Reading expressions of equal groups as "*and* stories" and as "'s stories"—"the long way" and the "short way."

(*b*) Reading numerical written exercises in + and 's.

(*c*) Reading graphic object work.

(*d*) Writing from oral dictations.

NOTE.—This section is also table work. No pupil should be permitted so to stand at the table that he must see and read objective work backward—from right to left. The teacher on one side of the table must construct from her right to her left so that the expressions will be from left to right for pupils who stand facing her. The habit of considering constructions always from left to right can not be gained by a pupil who stands where he must read or hear others read from his right toward his left. All pupils must stand opposite the constructions. This is of the utmost importance, and teachers must bear it in mind carefully.

(*a*) The Oral Reading or Interpretation of Objective Work.

The teacher constructs the expressions for the pupils to interpret.

The following is a lesson plan of the work of this section:

Lesson Plan.

(This illustrates an expression constructed by the teacher.)

"Read it." ("A four and a one.")

"Make the crayon say it." (4 + 1.)

"What is the first group?" "The second?"

"Are the groups alike?"

(Representing the teacher's construction.)

"Read it." ("A three and a three and a three and a three.")

"Make the crayon say it." (3 + 3 + 3 + 3.)

"What is the first group?" "The second?" "The third?" "The fourth?"

"Are the groups alike?"

"Read it the short way, then." ("Four threes.")

"Make the crayon say it the short way." (4 3's.)

"Read it." ("Three twos.")

"Make the crayon say it." (3 2's.)

"What is the first group?" "The second?"

"The third?"

"Are the groups alike?"

"Read it the short way." ("Three twos.")

"Read it the long way." ("Two and two and two.")

"Make the crayon say it the long way." (2 + 2 + 2.)

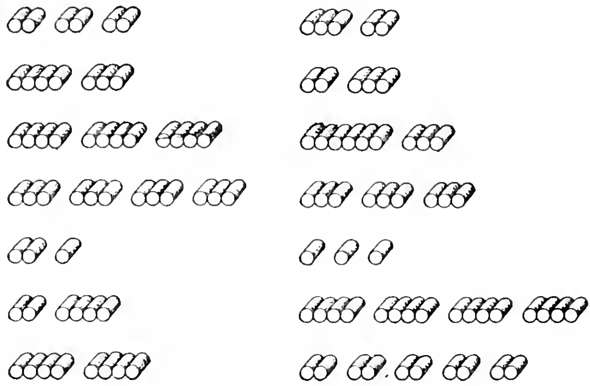
NOTE.—The teacher may perhaps make an equal group construction having in mind the short way interpretation. If the pupil interprets it the long way, it is, of course, correct. An interpretation which is true to the construction must be accepted, regardless of the intention of the construction. The other interpretation may be called for afterward.

(*b*) Reading Lessons in Written Work.

Written lessons like the following should be given from the blackboard, the purpose being simply to give practice in reading written work. Such exercises are reading lessons in number, there being no object work connected with it.

4+4+4	4+1	2 2's	4+2
3+4	1+4	4+2	4 2's
4+3	2 3's	2 4's	2+3
4 3's	3 2's	2+4	3+2

(*c*) Reading Lessons in Objective Work—using graphic object work on the blackboard. The following is an illustrative exercise:



NOTE.—In exercises of this kind the second reading of equal group expressions should be required in every case. When the pupil reads such an expression at first the long way, the teacher's suggestion to give the "short" reading should be preceded by the question "are all the groups alike?" The habit of looking to see if the groups are alike is very necessary in the development of power in the use of this number language. The pupils must have many of these exercises.

(*d*) Writing from Oral Dictations.

This is merely a language lesson in number. Numerical expressions are dictated orally for pupils to write on paper or blackboard.

NOTE.—The exercises outlined above under (*b*), (*c*), and (*d*) should be given daily until the completion of Section 8

5. Cards for Objective Reading Work.

Another form of reading lesson is in the use of cards in the hands of the teacher for sight work. Each card has a graphic objective expression on it, and these cards are held before the pupil one after another for rapid interpretation. This is similar to the use of word or phonic letter cards in primary reading. These objective expressions may be made of squares, circles, or any other uniform figure. The circle is to be preferred, because, if the figures are colored, as they may be to advantage, this form permits the units of a group to be tangent without the loss of their individuality to the pupil.

6. Oral Dictations for Pupils to Construct.

For this exercise the pupils are at their seats with a supply of objects. The teacher dictates expressions orally one by one for pupils to construct. It is very important that she inspect each construction as it is made. She must know that every pupil has the expression correctly constructed before the next is dictated.

Exercises of this kind, being given orally, are ear drills in number. The child is learning

to associate the concrete expression with the spoken. It is one of the most important of the exercises in the language section of the work. Many of these exercises must be given, the work paralleling that of Sections 7 and 8.

7. Written Dictations for Pupils to Express Objectively.

(*a*) Table Work.

The dictations are given on the blackboard one by one. Pupils construct, read the construction, then push back the objects. If an expression has equal groups, the second reading should also be required.

Note.—When the dictation of an equal group expression is in the "long" or + form, after the pupil has constructed and read it the question "Are the groups all alike?" should be asked, and following this there should be the request for the "short" reading.

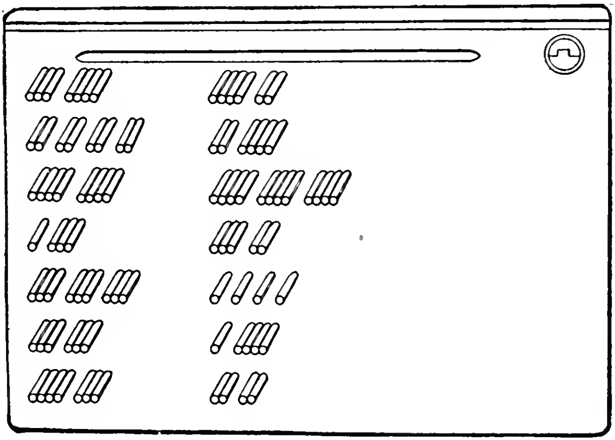
(*b*) Seat Construction Work—pupils at their desks.

The dictations are given one by one and the constructions made and read as in (*a*), but the dictations are not erased or the objects pushed aside. If an exercise of a dozen or more expressions is given, the dictations should remain on the blackboard so as to form two or more columns when completed.

The dictations will accumulate into a form as follows:

3 + 4	4 + 2
4 2's	2 + 4
2 4's	4 + 4 + 4
1 + 3	3 + 2
3 3's	4 1's
3 + 3	1 + 4
4 + 3	2 2's

The constructions on the pupil's desk should be in corresponding form when completed, as follows:



NOTE.—If pupils are trained to make these constructions on the left half of their desks, it will leave the right half clear for them to use in the written work which follows. See (*c*).

After a construction is made the teacher must examine it to see that it is true to the dictation—this before it is *read*. If a mistake is found in a child's work, it would not be proper to point out the nature of the mistake. It is enough to point to the expression which he has constructed incorrectly. The child should discover and rectify the error himself. This is applicable to written work also, as in (*c*) following. The child should be made to rely as much as possible upon himself.

(*c*) Seat Written Work.

After an exercise has been completed as in (*b*), the blackboard dictations should be erased and the pupil required to take paper and pencil and write the whole exercise from the construction work on his desk. In this work the pupil should *not* be required to write the equal-group expressions in *two* ways. Let him write each of such expressions once, using whichever form he pleases. The teacher must examine each pupil's written work to see that it is true to his construction work.

Note.—It should not be expected nor desired that the pupil's written work should be an exact reproduction of the original dictation, or that two pupils should agree in all the written expressions of an exercise. In the unequal-group expressions the written work will be the same, but an equal-group expression on the blackboard in the 's form may appear on a

child's paper in the + form, or a + expression of this kind may appear on a child's paper in the 's form. Variations of this kind must be accepted. Each child must be given full independence. The teacher's aim in examining the desk object or written work should be to see (1) that the object work is true to the blackboard dictation, and (2) that the written work of each individual is true to his desk object work.

CHAPTER II

SEAT WORK IN NUMBER LANGUAGE

8. **Aim**—to give the child practice in the use of the language which has been developed in the preceding chapter.

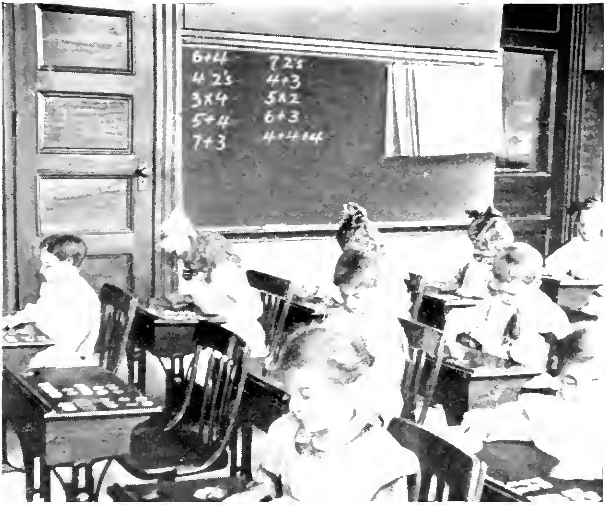
FORM OF WORK.—Each exercise to consist of three parts: (*a*) the blackboard dictation, (*b*) concrete expression by the pupil at his desk as "busy work," (*c*) writing the exercise on paper (also "busy work") from the object work on the desk—the blackboard dictation having been erased.

Teaching Plan.

The teacher writes on the blackboard a dictation exercise of ten, twenty, or more expressions for the child to construct on his desk without assistance or oversight. The whole exercise is written in advance, and the teacher is free for other school duties while the pupil is at work upon his constructions.

When all have finished the constructions, the teacher should examine the work of each to see that it is correct.

The dictation is now erased and the pupil



SEAT WORK IN NUMBER LANGUAGE.

(Section 8 (*a*), (*b*), and (*c*)).

A blackboard dictation in number language and pupils at work constructing the expressions with objects. The construction work completed, the teacher will erase the dictation and the pupils with pencil and paper will write the exercise from their object work. (See Section 8 (*c*), Note 1.)

is given paper and pencil to write the exercise from the object work on his desk.

The teacher is free for other work while the child is writing, but the child's object work must remain on his desk after the written work is done until the teacher has had an opportunity to compare it with his written work.

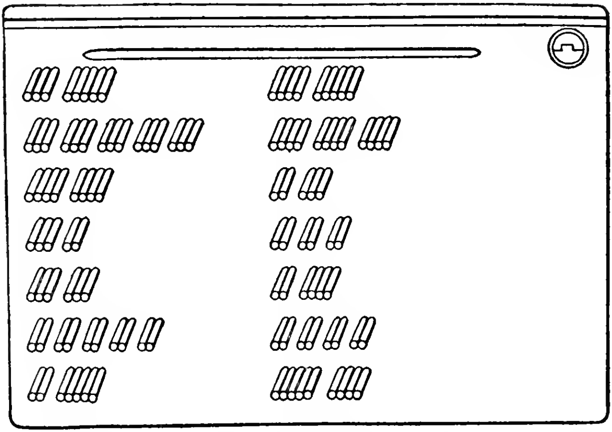
The following are given to illustrate the three steps in this work:

(*a*) The Teacher's Blackboard Dictation:

3 + 5	4 + 5
5 3's	3 4's
4 + 4	2 + 3
3 + 2*	3 2's*
2 3's	2 + 4
5 2's	4 2's
2 + 5	5 + 4

* The child in seat construction work of any kind from blackboard dictation reads the expression, constructs it, finds the next, constructs it, and so on. It would save the child's time and eyes to have landmarks of some kind at regular intervals in each dictation column to aid him in quickly "finding the place" when he looks up from a construction to find the next expression. It is suggested that when the dictation column consists of five expressions, the third be written with colored crayon; in a column of seven, as in this dictation, the 4th; in a column of eight, the 3d and 6th; etc.

(*b*) Pupil's Seat Construction Work.



(*c*) Pupil's Written Work from the Seat Construction.

After completing the construction work (*b*) the pupil is given paper and pencil, the blackboard *dictation* (*a*) *is erased*, and he is required to write the exercise from the object work on his desk.

The following illustrates what a pupil might write from his construction work:

3 + 5	4 + 5
3 + 3 + 3 + 3 + 3	4 + 4 + 4
2 4's	2 + 3
3 + 2	3 2's
2 3's	2 + 4
5 2's	2 + 2 + 2 + 2
2 + 5	5 + 4

Note.—The exercise is finished when this written work is completed and the teacher has compared it with the seat construction work. There is no recitation work, no questioning.

Note 2.—The work of this chapter, with that in Chapter III added at the proper time, must be continued day after day for two, three, or perhaps four weeks, until the pupils acquire a fair degree of accuracy both in construction and in writing from their objective work.

Note 3.—The Counting Material and its Use.

In the selection and use of the objects for the child's seat language and seat counting work of every kind in this Outline the following suggestions should be carefully observed:

(*a*) This counting material should consist of two or more classes of objects. An assortment compels the child to select equal units when constructing an expression. (See Section III (*g*).)

(*b*) The objects used should be of good quality, those of each kind—the lentils, the inch-sticks, the small pegs, the large pegs, etc.—being as nearly perfect and uniform as possible. The pupils should be taught to discard all imperfect objects of each kind whenever found. (See Sections III (*g*) and V (*c*).)

(*c*) Each concrete expression should be made of one class of objects throughout—pegs of one size, one kind of lentils, one kind of inch-sticks, etc. It is not necessary that the same class of units should be used in two successive concrete expressions in an exercise. One may be of a certain class of lentils, the next of large pegs, the next of small pegs, etc. Some teachers go so far as to require that the units of each expression shall be uniform also in *color*. This makes the *uniformity* requirement more emphatic, but it is not necessary mathematically.

CHAPTER III

THE INTRODUCTION OF $\times$

9. The Aim—to teach the child that there is another way to write 's stories—using $\times$.

Written expressions in $\times$ are to be read as if written with 's.

The introductory work must be very simple. We merely *tell* the child. He is taught that 4×3 means 4 3's, 5×3 is to be read 5 3's, 3 5's may be written 3×5. No other explanation. The "times" reading must not be introduced until later.

The work outlined in this chapter may be introduced soon after that in Chapter II is begun. The use of expressions in $\times$ will then become a part of the work of that chapter.

(*a*) Blackboard Reading Lessons, like those suggested in Section 4 (*b*), should be prepared with 's expressions written sometimes with 's and sometimes with $\times$. The aim of the exercises is to teach the child to read expressions with $\times$ as if they were written in the 's form. The following illustrates the form of such exercises:

3×4	5 6's	6×3
2 3's	6×5	$3 + 6$
$2 + 3$	$4 + 5$	5×1
$3 + 4$	5×4	$4 + 4$
4×3	$3 + 5$	4×2

(*b*) Oral Dictation Exercises should be given involving expressions in 's and + for pupils to write. Pupils should be required to write each of the 's expressions in this exercise in the two ways. For the dictation *two threes* the child should be required to write 2 3's, 2×3.

The dictations for the seat work outlined in Chapter II should be continued, but should now include expressions in which $\times$ is used.

10. Constructions in which the Pupil has a Choice.

This work is similar to that in Sections 8 and 9 except that it gives the child some choice in determining what each expression shall be. It involves but one teaching step—the development of the written language.

Teaching Plan.

The teacher writes on the board some elliptical + expression, such as 4 +.

"I want you to make this. It means 4 and anything else that you wish."

Individual pupils are then asked, "What did you make?" "What did you make?" etc.

$+6$. (Written by the teacher on the blackboard.)

"This means something and 6." "Make it." Individuals are then asked as before what they made.

$\times 3$. (Written on the blackboard as before.)

"This means as many 3's as you please." "Make them." "What did you make?" etc.

$4\times$. (Written on the blackboard.)

"This means 4 groups of any kind you choose." "Make them."

5's. (On the blackboard as before.)

"This means as many 5's as you choose." "Make them."

3 's. (On the blackboard as before.)

"This means 3 groups of your own choosing." "Make them."

Continue such work until pupils interpret and construct without suggestion.

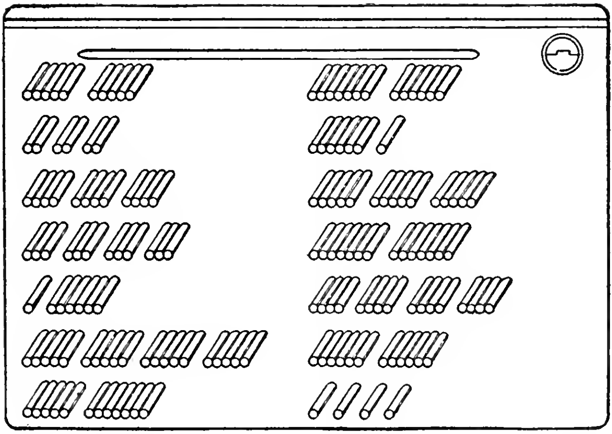
11. Exercises with Choice for Seat Work.

Blackboard dictations for seat "occupation work" should now be given in the following elliptical form, the child having an opportunity for the exercise of choice in each expression constructed:

(*a*) Blackboard Dictation.

5 +	+ 6
3 ×	6 +
× 4	3 's
3's	2 ×
+ 6	× 4
× 5	6's
5 +	4 ×

(*b*) Pupils' Seat Work.



In construction work from dictations without choice it is evident that the concrete work would be the same on all the desks, the written work alone showing variations. In construction work from these elliptical dictations, however, there may be as many va-

rieties of constructions from one dictation exercise as there are pupils in the class. This individuality makes the teacher's inspection work more difficult, but it leaves no doubt as to the child's knowledge of the work.

(*c*) Pupils' Written Work:

$5+5$	$7+6$
3×2	$6+1$
3 4's	3×5
4×3	2×7
$1+6$	4×4
4 5's	2×6
$5+7$	$1+1+1+1$

NOTE.—In this and all other language construction work the blackboard dictation should be erased before the pupils begin the written work.

CHAPTER IV

COUNTING WITH GROUP UNITS

AIM—to give the child the opportunity to apply the language which he has learned in the work of the preceding chapters in counting with group units in the + and 's (or ×) processes.

"WILL MAKE" LANGUAGE.

It will be necessary to teach the child a meaning for the sign =. The dictations will be in the form—6 =, 4 =, 8 =, 5 =, etc. The child must be taught to read such expressions —6 will make, 4 will make, 8 will make, etc., the meaning of the sign = being expressed by "will make." The pupil will understand by 6 = that he must take 6 objects and construct a group expression with them, 4 + 2 or 2 + 4 or 2 × 3 or 3 + 3 or any other possible expression that he may choose. After the construction the full expression (the equation) must be written. If from the dictation 6 = the child makes 4 + 2, the full written expression would be 6 = 4 + 2. The equation is read —six will make four and two.

OUTLINE OF THE WORK:

1st. The oral and written language of the equation.

2d. Exercises in counting.

Teaching Plan.

12. Development of the Oral and Written Equation Language.

The pupils at their desks, with objects, paper, and pencil. The teacher at her desk, with large objects.

(*a*) "Take 5 pegs." (Pegs or any other class of uniform objects may be used.)

"I will take 5, too, here on my desk." (The teacher uses inch cubes or other large objects that can be seen by all the pupils.)

"Make a story with your five—any story you choose. I will make one, too."

"What story did you make?" (This is asked of each pupil.) (Some pupils will have made $3+2$, others $2+3$, others $4+1$, others $1+4$, etc. There will be a variety of constructions. This is as it should be.)

"I made a 4 and a 1."

"I will write my whole story"—$5 = 4 + 1$. (Teacher reading*) "5 will make a 4 and a 1."

"This sign (=) means will make" (making this explanation to the pupils).

"I want each of you now to take a pencil and write *yours* on your paper."

(The teacher must see that each writes his expression correctly.)

"You may read your story." (This should be done by all.)

(*b*) "Take 4. I will take 4, too."

"Make a story. I will make one, too."

"What is your story?" (Asked of each as before.)

"I made 2 2's."

"I will make my story"—$4 = 2 \times 2$, "4 will make two twos."† (The teacher reads as before while writing.)

---

* This reading should be done while writing, the teacher speaking each word as she writes—the word *five*, the expression *will make*, the *four*, the word *and* (+), and the *one* being spoken as the crayon forms the corresponding character.

† There is no word to go with the $\times$ in the reading of expressions of this kind at present. In talking while writing (see Note * under (*a*) above) the one word *twos* must be spoken while writing the two characters "$\times$ 2".

"I want each of you to write his story on his paper."

"You may read your story."

(*c*) 8 = (Written by the teacher on the blackboard in the column.)

"When I write this it means that you are to take so many pegs and make something without my saying anything."

The teacher asks each what he has made, then pupils read and write their concrete expressions as before.

The teacher announces her own construction and completes her written expression afterwards — $8 = 2 \times 4$, (reading as she writes) "8 will make two fours."

(*d*) 7 = (Written on blackboard in the column.)

Write the expressions used in this part of the section ((*a*), (*b*), (*c*), etc.) on the blackboard under one another so that the dictations at the close of an exercise will be in one or more columns. This column arrangement of the dictation and the pupil's concrete work, also preserved on his desk in column form, is to prepare the child for the form of work used in "seat work" below. Here the dictations *accumulate* into columns; there they are given in that form. The child also learns to arrange his seat work according to the dictation.

"What does this tell you?" ("It tells me to take 7 and make something.")

"You may do so."

"What is your story?" (Asked of each pupil.)

"Write your story."

"My story is $5+2$." The teacher now completes her written expression on the blackboard—$7=5+2$, (reading) "7 will make $5+2$."

(*e*) "I am now going to give you three stories to make and write. I want you to make them all and then write them. I'll not make them at all. I want to see you do the work without my saying anything." (These should form a separate column.)

$6=$

$4=$

$7=$

NOTE.—While the pupils are making their constructions, the teacher gives attention to individuals to see that every child understands what he is to do. When all have finished she should ask individuals to read their "stories."

(*f*) "I am now going to give you five stories to make and write." (Let these form a separate column.)

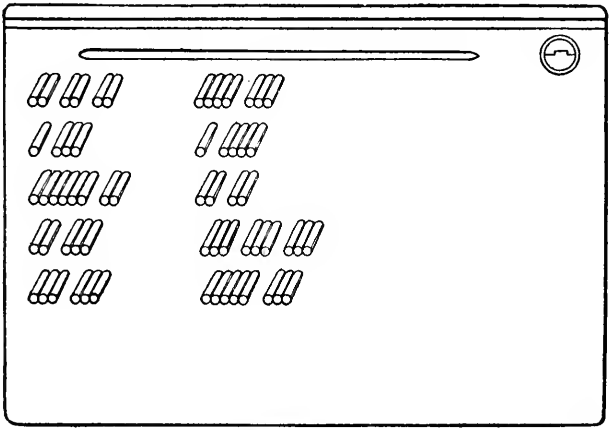
7 =
8 =
6 =
5 =
4 = (See note under (*e*) above.)

Seat Work in Counting.

13. (*a*) **Form of the Blackboard Dictation.**

6 =	7 =
4 =	5 =
*8 =	*4 =
5 =	9 =
6 =	8 =

(*b*) Form of Pupil's Seat Construction Work. This illustrates what a pupil might construct from (*a*).



* See Section 8 (*a*), note.

NOTE.—This construction work should be completed before the written work (*c*) is begun. The dictation may or may not be erased; the pupil can get no improper assistance from it.

(*c*) Form of Pupil's Seat Written Work.

$6=2+2+2$	$7=4+3$
$4=1+3$	$5=1+4$
$8=6+2$	$4=2\times 2$
$5=2+3$	$9=3+3+3$
$6=2\times 3$	$8=5+3$

NOTE.—The teacher must examine the work of each pupil to see that his written work (*c*) is true to his constructions (*b*).

14. Recitations.

These exercises have no recitation or drill work connected with them. When the child has made his constructions and written them out and the work has been inspected by the teacher, the dictation has served its full purpose. There is no recitation work until we come to "memory tests" in Chapter V.

15. Special Dictations.

If pupils in their written work (*c*) are found to be using the + form of expression to the exclusion of the × form, or the reverse, with

expressions where the groups are equal, the dictations may be given in the following form

+	×	+
5 =	6 =	8 =
6 =	8 =	7 =
8 =	9 =	9 =
4 =	4 =	4 =
7 =	6 =	5 =

The sign over the column indicates to the child the character of the constructions to be made and the form in which to express them. The above dictation would call for equal-group constructions for the second column to be expressed in × form. Some of the constructions in the first and third columns may have equal groups, but they must be written the "long way" (as "and" stories). The dictations should not be thus marked except when absolutely necessary to force the use of both + and × expressions in proper proportion in equal group readings. If continued for a period of one or two weeks, it should then be discontinued until necessity calls for it again. It limits the child's freedom of choice and is indefensible except for the purposes stated.

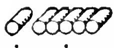
16. Time to be Given to this Work.

The seat work as outlined above must be continued day after day and week after week until the "memory tests" in Chapter V indicate that memory has taken complete hold upon the facts.

17. Assistance not Required.

This seat work does not require the presence or assistance of the teacher. She is free for other school duties while pupils are at work upon it. Her duty is simply to inspect the work after it is written. The object work must remain upon the pupil's desk until this inspection has been made.

18. The Construction of Ones.

There will be a tendency to the construction of groups of 1's. For a dictation like 6 = in a column calling for + stories pupils will be found making  or , or . The last two are not objectionable if they are not in too large proportion. The first should be discouraged whenever there is a settled tendency to make it. The work has too little value in it to permit waste of time in construction and writing. The same may

be said of columns calling for $\times$ expressions. $8 = 8 \times 1$ is not valuable as a memory product.

19. Variety in Constructions.

In order to compel greater variety in constructions—to force the child to construct more than one of the stories that 6 will make, or 8 will make, etc., and thus get broader experiences—dictations may take the following form, the same number to be repeated in succession not more than three times:

8 =	9 =	6 =
8 =	7 =	6 =
5 =	7 =	8 =
5 =	7 =	8 =
6 =	4 =	4 =
6 =	9 =	9 =
6 =	9 =	7 =

The pupil should understand by this that when a number is given two or three times in succession he must make different stories. If the same number is again given but with other numbers intervening, he should be free when he comes to it to make whatever he chooses, regardless of his previous constructions. In the dictation given above there is a group of three successive 6's in the first column

and a group of two in the third. The child in constructing the latter of these groups should not be required to consider the expressions constructed in the former.

20. "And" Expressions of More than Two Unequal Terms.

Any tendency to construct expressions like $4+2+3$, or $1+4+2$, etc., must be discouraged. The memory product of such work is not fundamental. $4+2+3$ is not an expression to be memorized. To one mind it is $6+3$, to another $4+5$. In either case the "fact" in memory is an expression of two terms. Combinations of more than two unequal terms belong to the purely oral period of number work—they are brought into use when the "reckoning" period begins. The child who knows $3+4$ and $7+2$, does not require objective work, or objective illustration, or even instruction, assistance, or suggestion of any kind to know $4+3+2$.

CHAPTER V

MEMORY TESTS

21. After two or three weeks of the seat work outlined in Sections 13–20 the teacher should begin to question each individual of the class as to what he remembers of the "will make" work. Thus far the teacher has dealt with the class as a whole or with groups except when examining seat work. In these memory tests, however, she must deal with the individual pupil. Each should be questioned as frequently as possible—every day or two.

22. Form of Test.

The question form is What will 4 make? What will 6 make? What will 3 make? etc. If the pupil does not respond to the question *at once*—does not remember what the number will make—the question must be dropped immediately. If the child does not remember what 4 will make, or 5 will make, etc., it indicates that he has not had sufficient experience with that number in his seat work. The "will make" work must be continued for a few days longer and the test be made again. Under

no circumstances should he be made to think that he has lost prestige by not being able to answer a question of this kind, neither should he be coached nor drilled on the number facts. What the child is to remember should come to him in the natural way—through construction experiences.

This number work is based upon the principle that knowledge of number facts should come as the product of number experience in the seat work. If the child does not have knowledge of any given number fact, the teacher must postpone the question until he has had more seat work. Children differ very greatly in the memory product of a given amount of experience. One child after a week or two of the work in Sections 12 and 13 will remember that 4 will make 2 2's, $3+1$, $1+3$, and $2+2$, or at least one or more of the facts, while others may remember nothing.

The teacher must discover in the case of every member of her class when memory begins to take hold of experience. She should know when there is one remembered fact, when two, when three, etc. Each should have frequent opportunity to tell what he remembers of what 3 will make, 5 will make, 4 will make, etc.—every day, if possible. If

one child remembers many facts he should be given a chance to tell them. If another remembers but one he should be called upon to tell that.

23. Pupils to Learn all the Facts.

It is necessary that the pupil know all the facts of each number to 10. To illustrate, 6 will make $1+5$, $2+4$, $3+3$, $4+2$, $5+1$, 3×2, and 2×3; 5 will make $1+4$, $2+3$, $3+2$, $4+1$; etc. The child will get all these from the experience of two or three months of daily seat work, if—

1. There is frequent opportunity for him to tell what he remembers.

2. The recitations are so conducted that the facts become properly arranged in the mind.

24. Orderly Arrangement through Recitation.

The pupil must be discouraged from giving facts in regular ascending or descending order, as—6 will make $1+5$, $2+4$, $3+3$, etc., or 6 will make $5+1$, $4+2$, $3+3$, etc. To this end the teacher must not ask the child to give all the things that 4 will make, or all the things that 7 will make, or all the things that 5 will make, etc.

The reason for this is that if asked for all

the facts of a number he *naturally* takes them in ascending and descending order. Such a recitation would be orderly in its arrangement of facts, but the listener could not know whether the child is telling of remembered experiences or simply counting.

The proper arrangement of number facts in the mind has some elements of order and some of disorder. If it is found that the child knows that 6 will make 2 3's and also that 6 will make 3 2's, he should be led to put the two facts together. Asked what 6 will make, if he should give one of the pair he should at once without suggestion give the other—6 will make 3 2's and 2 3's, or 6 will make 2 3's and 3 2's. The same may be said of other opposites—4+2 and 2+4, 4+5 and 5+4, 4 2's and 2 4's, etc. When he remembers both of two opposites, the one should follow the other in recitation. This may be brought about by development. A simple plan is merely to give strong approval when a child thinks of two stories that "go together." He will discover what pleases, and his desire for approbation will in time get him into the habit of thus arranging the number facts. Beyond this grouping of opposites, order in arrangement is neither desirable nor safe.

25. The Use of Ones.

Under "The Construction of Ones," Section 18, the suggestion was made that the construction of 1's (5 will make $1+1+1+1+1$, or 5 will make 5×1, etc.) should be checked whenever there is a settled tendency to make expressions of that kind. A few such constructions are permissible for the sort of dreamy pleasure the child gets out of it. In the memory work pleasure is a secondary consideration, and from the very first there should be a judicious disapproval of these "little easy stories," for the "big stories" like $3+2$ and $2+3$, when asked what 5 will make, or 3×2 and 2×3, $4+2$ and $2+4$ in telling what 6 will make.

NOTE.—The expression 1×5 is artificial and bad at this period of the work. If the teacher does not make use of it the pupil will not think of it. The mind takes hold of *a* 5 or *a* 4, etc., but oral *one four* or written 1×4, etc., are abstractions at this stage of the work. They occur naturally under "has how many" in Sections 40, 41, and 44.

26. Modes of Testing Memory.

Some pupils on being asked to tell what a number will make will be able to give two or more "stories," others will give but one, a

second or third being given only in response to questioning.

The following is an illustration of an extreme case of the latter—the child not giving more than one "will make" fact or one pair of such facts without being urged by a suggestion or question. It has been suggested before that memory tests are inadmissible if answers do not promptly follow questions; therefore it is understood that in this there is no delay between the teacher's prompting and the child's answer:

"What will 8 make?" ("8 will make $6+2$ and $2+6$.")

"Can you think of another?" ("8 will make $7+1$ and $1+7$.")

"I do not like your little *one* stories. I want *big* ones. I want something with a 4 in it." ("8 will make $4+4$.")

"Can you not think of something with a 3 in it?" ("8 will make $3+5$ and $5+3$.")

"Can you give me a 's story?" ("8 will make 2×4 and 4×2.")

CHAPTER VI

"TAKE AWAY"

27. The Development of Subtraction Counting.

(*a*) Oral and Written Language.

The introductory work is best done with pupils at the table in groups of convenient size for work, there being on the table a plentiful supply of objects—large objects, such as inch cubes, large splints, or both.

Development Plan.

"Let us all take 6."

"Now let us take away 2."

(Teacher and pupils each take away the 2, putting them entirely away into the reserve supply on the table.)

"How many are left?"

"Take 8."

"Take away 4."

"How many are left?"

"This is the way the crayon says it"—

(8−4=4).

NOTE.—When the teacher "makes the crayon say it" in this and the other expressions in this development, *she should "talk" while writing*—speaking each

word as the crayon forms its sign—"8 take away 4 leaves 4." "Take away" is spoken with the $-$, "leaves" with the $=$. When the child writes such an expression on the blackboard for the class during this development work, he too should "talk as he writes."

"Take 7."
"Take away 6."
"How many are left?"
"This is the way the crayon says it"—$(7-6=1)$.

(The teacher, of course, "talks" as she writes; see note above.)

Questions to be asked on the written expression:

What tells how many to take?
What tells how many are left?
What tells how many to take away?

Questions should be at times in this form:

What says take away?
What says leaves?
What does 7 tell? (How many to take.)
What does 6 tell? (How many to take away.)
What does 1 tell? (How many are left.)

"Take 9."
"Take away 6."

"How many are left?"

"Who will write this whole story for me?"

(The pupil "talks" as he writes; see note above. Questions should be asked as above.)

(*b*) Reading Exercises.

$8-4$	$4-1=3$
$7-6$	$5-4=1$
$9-2$	$3-2=1$
$7-3$	$6-2=4$
$8-7$	$5-2=3$

(Exercises like the above are written on the blackboard for pupils to read. Questions like those above should be used with these reading exercises.)

(*c*) Writing Exercises.

(Oral dictations, pupils writing as teacher dictates. Pupils at their desks with paper and pencil.)

"7 take away 5." ($7-5$.)
"9 take away 7." ($9-7$.)
"8 take away 3." ($8-3$.)
"4 take away 1 leaves 3." ($4-1=3$.)
"8 take away 7 leaves 1." ($8-7=1$.)

(*d*) Form of Seat Work.

NOTE.—The teacher writes the expressions on the blackboard one by one, the pupils making the constructions. The dictations are in the form $6-3=$.

It will be necessary at the outset to give the pupils a form of construction that will show the group taken away in each construction. A plan must be adopted for putting these "taken away" groups where each may afterwards be found (1) by the pupil when he comes to write from his constructions, (2) by the teacher when she examines the seat object work. There are two ways in which these groups may be shown—(1) at the right of the "left" group at right angles to it

(7 − 3 ),

or (2) *above* the "left" group at right angles to it

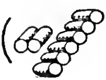
(7 − 3 ).

The former will be used in this outline.

As the manner of placing the object groups to show the "take away" expression is purely arbitrary, the teacher merely instructs in the form in which the "left" group and the "taken away" group are arranged.

Teach to arrange as follows:

7 − 3 = ()

8 − 6 = ()

9 − 4 = ()

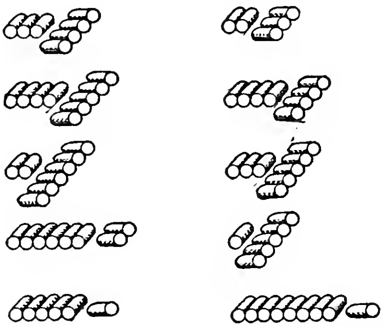
Continue this until the pupils have the form of seat construction. After each of these con-

structions ask them to write the whole story—

$$7-3=4$$
$$8-6=2$$
$$9-4=5$$

(*e*) Reading Lessons.

The teacher puts graphic objective reading lessons on the blackboard for pupils to read. This training is to enable them to read their seat constructions after the dictations are erased.



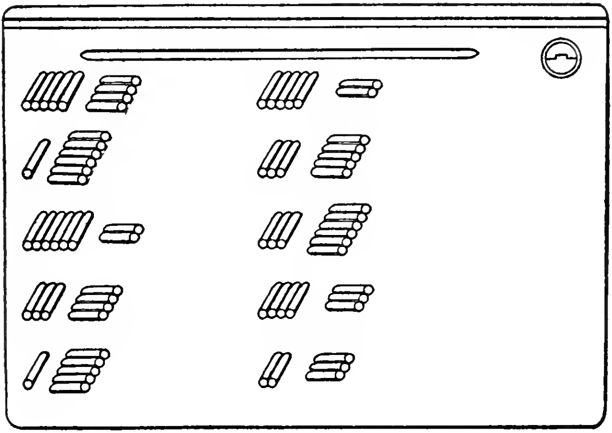
28. Counting in "Take Away" Form.

This gives the child "occupation work" in which he must make use of the language acquired in the work of Section 27.

(*a*) Blackboard Dictation

$9-4=$	$7-2=$
$7-6=$	$8-5=$
$8-2=$	$9-6=$
$7-4=$	$7-3=$
$6-5=$	$5-3=$

(*b*) Seat Construction.



(*c*) Seat Written Work.

$9-4=5$	$7-2=5$
$7-6=1$	$8-5=3$
$8-2=6$	$9-6=3$
$7-4=3$	$7-3=4$
$6-5=1$	$5-3=2$

NOTE.—In a "take away" exercise like this the dictation should be erased before the child begins to write from his construction work. After a few exercises in "take away" like the above the teacher should cease giving exercises consisting wholly of subtraction expressions. The dictations should be "take away" and "will make" mixed.

(*d*) Blackboard Dictation—Mixed.

Exercises of this kind give counting in all the forms learned.

$9=$	$6=$
$7=$	$6-4=$
$7-5=$	$7=$
$8=$	$7=$
$9-5=$	$8-5=$
$8-6=$	$9-4=$
$5=$	$8=$

29. "Take Away," with Choice.

NOTE.—The "take away" work as outlined in Sections 27 and 28 gives the child no choice in his constructions, and is thus wanting in one of the elements which lend interest to the work after the novelty of the language wears off. Its use should be continued only until the child acquires a working knowledge of the "take away" language, when the following form of dictation, which affords a choice in each construction, should be developed and substituted:

(*a*) The Written Language—Choice in the Subtrahend.

The only development necessary is the language of the written *dictations*.

Development Plan.

Pupils at their desks with objects.

The teacher writes on the blackboard:

(1) $6 - \quad =$

"This means that you are to take 6 and take away something,—take whatever you please."

"You may each take 6."

"Take away something."

"What did you" (indicating some pupil) "take away?" ("Four.")

"You may write your whole story." ($6 - 4 = 2$.)

"What did you" (indicating another pupil) "take away?" ("Three.")

"You may write your whole story." ($6 - 3 = 3$.)

(2) $7 - \quad =$

"This means that you are to take 7 and take away something."

"You may do so."

"What did you take away?"

"You may write your whole story."

"What did you take away?"

"You may write," etc.

(*b*) Seat Construction Work—Choice in the Subtrahend.

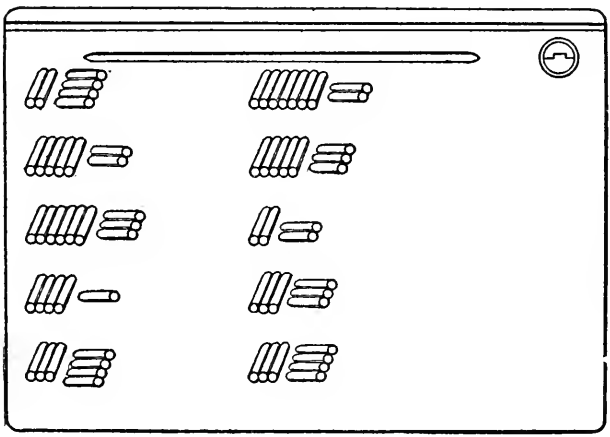
After three or four exercises like the above it will be possible to give, for seat work, exercises with choice in the following forms:

(1) Blackboard Dictation.

$6 - \quad =$	$9 - \quad =$
$7 - \quad =$	$8 - \quad =$
$9 - \quad =$	$4 - \quad =$
$5 - \quad =$	$6 - \quad =$
$7 - \quad =$	$7 - \quad =$

(2) Pupil's Desk Construction.

This shows a construction that some pupil may make from the dictation.



(3) Pupil's Desk Written Work.

$6-4=2$	$9-2=7$
$7-2=5$	$8-3=5$
$9-3=6$	$4-2=2$
$5-1=4$	$6-3=3$
$7-4=3$	$7-4=3$

(*c*) The Written Language—Choice in the Minuend.

Development Plan.

Pupils at their desks as before, with objects.

The teacher writes on the blackboard:

(1) $\quad -4=$

"This means that you are to take some objects—as many as you please—and take away 4."

"Each of you may take some objects."

"How many did you" (indicating a pupil) "take?"

"How many did you" (indicating another) "take?"

"How many did you" (indicating another) "take?"

Note.—Questions like the above should be asked of many of the pupils. A pupil may be found who has taken less than 4. The question should then be asked, "Can you take away 4 if you have only 3?" To get the pupil to realize in advance that it will be necessary to take at least as many objects as the number to be taken away will require time and patience. The best and quickest way to accomplish the result is to let him puzzle and worry over each construction (the teacher waiting in silence) until he finds the solution for himself. The only assistance

permissible is the question here suggested. After all have completed the construction, one individual and then another is asked to write his whole story on the blackboard.

(2) $-6=$

"This means that you are to take as many objects as you please and take away 6."

"Each of you may take some objects."

"How many did you take?"

"How many did you take?" etc., as in (1).

"You may write your whole story on the blackboard." (This to several, one after another.)

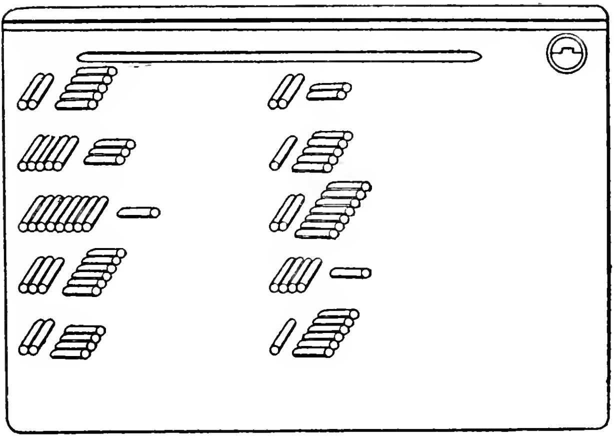
This plan is followed until pupils understand the written language and until they know that it will be necessary to take at least *as many* objects as the number to be taken away.

(*d*) Seat Construction Work—Choice in the Minuend.

(1) Blackboard Dictation.

$-5=$	$-2=$
$-3=$	$-5=$
$-1=$	$-7=$
$-6=$	$-1=$
$-4=$	$-6=$

(2) Pupil's Desk Construction.



This shows a possible construction from the dictation.

(3) Pupil's Desk Written Work.

$7-5=2$	$4-2=2$
$8-3=5$	$6-5=1$
$9-1=8$	$9-7=2$
$9-6=3$	$5-1=4$
$6-4=2$	$7-6=1$

(*e*) Seat Construction Work—Choice Mixed. Work should now be given in which the dictations are combinations of the choice work of (*b*) and (*d*), as follows:

(1) Blackboard Dictation.

6 − =	9 − =
7 − =	− 3 =
− 4 =	− 2 =
− 3 =	5 − =
8 − =	7 − =

(*f*) "Will Make" with "Take Away."

The "will make" work of Section 13 should now be resumed but combined with "take away" with *choice*.

The dictations for the seat construction and seat written work will take the following form:

(1) Blackboard Dictation.

6 =	7 − =
8 =	7 =
8 − =	− 4 =
7 =	− 3 =
− 2 =	9 =
− 4 =	8 =
9 =	8 =

The dictations may be in the above form (1), or they may be given with the "will make" and the "take away" in separate columns, as follows:

8 =	− 4 =	8 =
4 =	− 3 =	8 =
7 =	− 5 =	6 =
7 =	8 − =	6 =
9 =	7 − =	7 =
9 =	9 − =	5 =

CHAPTER VII

"TIMES"

30. This word requires no development. It should be given merely as another way of expressing 's stories. The $\times$ says *times*. 3×7 may be read 3 7's or 3 times 7.

3 times 7 means 3 7's.

Teaching Plan.

(*a*) The Oral Language.

3×2 (written on blackboard by teacher).

"Read it." ("Three twos.")

"There is another way to read it—3 times 2."

(In giving this reading the teacher points to each part as she reads it—to the $\times$ when she says *times*.)

4×3

"Read it." ("Four threes.")

"Read it the other way." ("Four times three.")

(The pupil should point to each character in the expression as he reads it.)

(*b*) Reading Exercises.

Exercises of this kind should be written on the blackboard for pupils to read, using both

the "times" and the "'s" readings, the pupils being asked to give the two readings for each expression.

3×3	3×5	7×2
4×6	5×3	2×7
6×2	6×8	7×6
2×3	8×7	6×5
3×4	7×8	5×6

(*c*) Writing Exercises.

To fix the word "times" in the child's working vocabualry, oral dictations should be given one by one for pupils to write at their desks or on the blackboard. Both the 's and the "times" forms should be used in such exercises. The following illustrate (1) the dictation and (2) the pupil's work:

(1) Oral Dictation.

Four twos.	Six times seven.
Three Fours.	Seven times four.
Four times three	Eight twos.
Two sixes.	Two times eight.
Two times seven.	Eight times eight.

(2) Pupil's Written Work.

4×2	6×7
3×4	7×4
4×3	8×2
2×6	2×8
2×7	8×8

NOTE.—Pupils will doubtless adopt the "times" reading for expressions of this kind. If so, the teacher should adhere to the 's reading. The purpose of this is to keep alive both forms. The only way to preserve these as interchangeable expressions is to see that they are interchanged in daily use.

31. Memory Tests.

The oral memory work suggested in Sections 21 and 22 must include tests in "take away" very soon after the seat work "with choice" has been begun. The form of questioning in "take away" is direct—Eight take away 3? Seven take away 5? Six take away 2? etc.

32. The Meaning of "Times".

The word has been introduced as an arithmetical idiom—no attempt having been made to develop its significance. The reason for this is that etymologically expressions in × are ambiguous. With some writers 3 times 4 means 3 4 times, with others it means 4 3 times. Etymology does not choose between them. Whichever interpretation is used it must be taken arbitrarily or for some reason apart from analysis of the language.

This number outline throughout will use the expression with the second of the above

meanings. This choice grows out of the object work. To connect multiplication with the child's previous language experiences, the expression at the beginning was necessarily in the form 3 4's. As this was natural language, the connection was made without effort. To extend this connection, 3×4 became another way of writing 3 4's. The step now being taken compels us, in order to continue the connection, to make 3 times 4 another way of saying 3 4's. Hence our interpretation of 3 times 4 as meaning 4 3 times.

33. "Multiplied by."

This expression should not be brought into use before the second or third year of number training. The variety of forms in which the root word occurs—multiply, multiplies, multiplication, multiplying, etc.—makes it impossible to avoid an interpretation based pon its etymology. We call "times" an idiom and give it an arbitrary meaning because its etymology fails to guide us. The etymology of "multiplied by," however, is more suggestive. 3 multiplied by 4 seems to mean 3 increased four-fold. Expressed mathematically this is 4×3. This is the meaning that object work requires.

CHAPTER VIII

SEAT WORK—WITHOUT OBJECTS

AIM—seat memory work in exercises in "and," "times," and "take away."

34. As soon as the oral memory work on the Outline to Section 32 is strong, the pupils should be given "will make" exercises to be written out without the object work. The blackboard dictations for these exercises are the same as have been given heretofore for construction work. In the exercises here contemplated the child writes his seat written work directly from the dictation without concrete work. In the *oral* memory work the teacher dealing with each individual separately needs no class divisions. A question that a pupil does not answer immediately is withdrawn. This *written* memory work, however, requires a division of the class at first, as the work can be given only to pupils *who have all the "will make" memory work* in "and," "times," and "take away" to 10. There will be at first a small section of the class able to do this work. One by one the pupils in the lower section will be put into

this division as they complete all the memory work to 10 in the oral tests. This seat work without objects is a great stimulus to all, as it makes a new and higher stage of work. Pupils who are given this work without objects feel that they have been "promoted." It stimulates the others to greater effort. Preparing separate blackboard dictations for the two sections adds dignity to the new work.

NOTE.—When this work includes "take away" exercises, the dictations must not be given in the "with choice" form.

CHAPTER IX

"HAS HOW MANY" (DIVISION)

SUMMARY OF TEACHING STEPS:

(*a*) Even Division.

1. Oral and written language.
2. Seat counting in division.
3. Seat counting in "will make" and even division mixed.

(*b*) Uneven Division.

1. Oral and written language.
2. Seat counting in uneven division.
3. Uneven division with choice.
4. Seat counting in "will make" and division mixed.

(*a*) "HAS HOW MANY"—EVEN DIVISION.

Meaning of Division.—The term division as used in this Outline is applied to that form of measuring in which the group to be measured is separated into groups of a given value. For illustration: $15 \div 5 =$ will mean that 15 is to be made into groups of 5 units each. In applied problem work later this meaning must be strictly adhered to, the child being required, in written analyses involving group-

ing processes, to discriminate between the measurements which are division and those which are partition (Sec. 76). Accurate language, clear imagery, and in fact objective work itself in general division are impossible where this discrimination is not strictly made.

1. Oral and Written Language.

NOTE.—The child has now become so familiar with number construction and language that this development, which is very simple, may be made with the whole class at one time at their desks if the teacher so prefer. Whatever the class arrangement, the teacher must know what response each child makes to each development step.

Development Plan.

35. Oral and Written "Has How Many."

"Take 8."

"Make it into 4's." ()

"How many 4's?" ("Two.")

"Take 6."

"You may find how many 2's it has."

()

"I will make the crayon tell us how to take 6 and find how many 2's it has"—(6 ÷ 2)—(the teacher "talking" while writing). (The three characters in ÷ may be read from upper

Has
dot to lower how   The sign = may be read
many?
"It
has." The expression $6 \div 2 = 3$ would thus
has
be read, "Six how twos in it? It has three.")
many

"Take 9."

"You may find how many 3's it has."

"This is the way the crayon tells us to take 9 and find how many 3's it has" (the teacher "talking" while she writes it)—($9 \div 3$).

"How many 3's did it have?" ("Three.")

"Now I will make the crayon tell the whole story" (talking)—($9 \div 3 = 3$).

"Take 8."

"You may find how many 4's it has."

"How many 4's did it have?" ("Two.")

"I will write the whole story"—($8 \div 4 = 2$).

"What tells us how many to take?" (8.)

"What tells us how many to have in a bundle?" (4.)

"What says 'has how many?'"

36. Reading Lesson.

$8 \div 2$	$6 \div 3$	$9 \div 3$
$10 \div 5$	$8 \div 2$	$8 \div 4$
$10 \div 2 = 5$	$6 \div 2 = 3$	$8 \div 2 = 4$

NOTE.—An exercise like this is written on the blackboard for pupils to read, the child going to the board and pointing to each character as he reads,—

has	has	has
8 how 2's?	10 how 5's?	10 how 2's?
many	many	many

It has 5, etc.

Questions like the following should be asked on each expression:

What tells us what to take?

What tells us how many to have in a bundle?

What says "has how many?"

After questions of this kind on three or four expressions, use another form of question:

"What does this tell" (pointing to one of the terms of an expression)?

"What does this tell?"

"What does this tell?" etc.; the answers being, of course, "It tells how many to have in a bundle," or "It tells how many to take," etc.

37. Writing Lessons—from Oral Dictation.

The teacher dictates expressions like those on Section 36 for pupils to write on paper or blackboard.

38. Constructions from Written Dictation.

$8 \div 4$ (written on the blackboard by the teacher).

"You may make this on your desks."

()

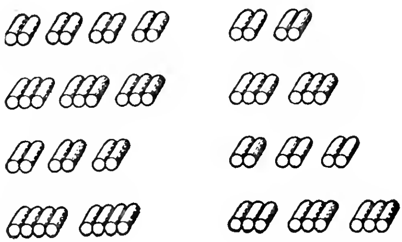
(The teacher must see every desk.)

"Take your pencils and write your whole story." ($8 \div 4 = 2$.)

NOTE.—After the expression has been written on the papers, one or more of the pupils should be asked to write it on the blackboard. After constructing and writing two or three expressions in this way the work of the next section may be begun. Frequent use should be made of the questions suggested in Section 36.

39. Graphic Object Reading Lessons.

The teacher writes graphic object work in "has how many" on the blackboard for pupils to put into oral language:



Exercises of this kind are necessary to prepare pupils to interpret their desk object work after the dictation is erased.

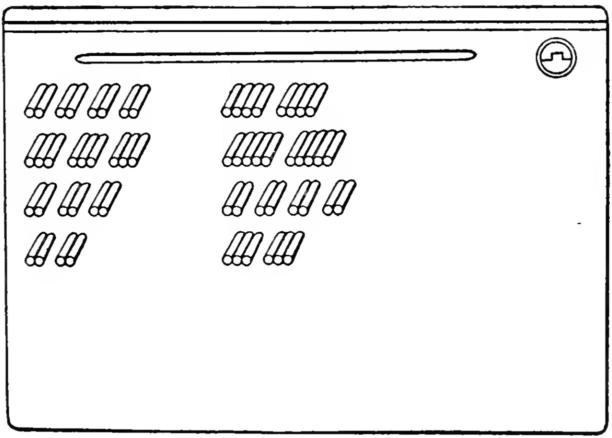
2. Seat Counting in Division.

40. Exercises of this kind give the child an opportunity to use the division language that has now been developed.

(1) Blackboard Dictation.

$8 \div 2 =$	$8 \div 4 =$
$9 \div 3 =$	$10 \div 5 =$
$6 \div 2 =$	$8 \div 2 =$
$4 \div 2 =$	$6 \div 3 =$

(2) Pupil's Seat Construction.



(3) Pupil's Seat Written Work.

The blackboard dictation should be erased before the written work begins.

8 ÷ 2 = 4	8 ÷ 4 = 2
9 ÷ 3 = 3	10 ÷ 5 = 2
6 ÷ 2 = 3	8 ÷ 2 = 4
4 ÷ 2 = 2	6 ÷ 3 = 2

41. Seat Counting in "Will Make" and Division Mixed.

Exercises of this kind should be given for several days—exercises which involve all the forms of counting that have been developed.

Blackboard Dictation.

8 ÷ 4 =	9 =	9 =
9 ÷ 3 =	9 =	8 =
6 ÷ 2 =	7 − =	8 =
10 ÷ 5 =	6 − =	− 6 =
6 ÷ 3 =	− 6 =	− 4 =
8 ÷ 2 =	8 =	6 =
10 ÷ 2 =	8 =	9 =

Note.—If mixed exercises of this kind are given for object work, the "has how many" dictations must be in separate columns; otherwise the child in writing from his seat constructions can not tell a "will make" from a "has how many" construction, there being nothing in the object work to distinguish the one from the other. The objective expression,

might come from $6 =$ or from $6 \div 3 =$. Again if the pupils at this time are in the memory period of the "will make" work (Section 34), the mixed work could not be given. The two forms of exercises would have to be entirely separated. This mixed dictation is to suggest the form of work to be given if the "will make" work is still in the construction stage.

(*b*) "Has how Many"—Uneven Division.

1. The Oral and Written Language.

Development Plan.

42. The Oral Language and the Concrete Expression Developed.

$8 \div 3 =$ (written by teacher on blackboard).

"You may make this." (Do not break in on their mental troubles for a moment. Let them puzzle over the two extra objects for a few seconds.)

"How many whole bundles?" ("Two.")

"Are there any left over toward another bundle?"

"How many?" ("Two.")

"You may place your part bundle beside your whole bundles" (teacher showing each). ()

"I will make the crayon tell the whole story—about the *whole* bundles and about the part bundle"—($8 \div 3 = 2\frac{2}{3}$).

NOTE.—The "talk" while writing this is, "Eight has how many threes? It has two, and two left over toward making another bundle of 3." The "two left over" goes with the 2 of the fraction and the "bundle of 3" goes with the 3 of the fraction.

$9 \div 4 =$ (written on the blackboard).
"You may make this."
"Do you have objects left over?"
"Place your part bundle beside the others" (teacher shows each). ()
"How many whole bundles?" ("Two.")
"How many left over toward making another bundle of 4?" ("One.")
"I will make the crayon tell the whole story"—($9 \div 4 = 2\frac{1}{4}$). ("9 has how many 4's in it? 2, and 1 left over toward making another bundle of 4".)
$7 \div 5 =$ (on the blackboard).
"You may make this." () (Teacher must see every desk.)
"How many whole bundles?"
"How many left over toward making another bundle of 5?"
"I want some one to write the whole story and 'talk'."

"You may write it on the blackboard" (designating some pupil).

$10 \div 4 =$

"You may make this." ()

"You" (designating some pupil) "may tell the whole story, taking what I wrote and what you have on your desk." ("10 has how many 4's in it? It has 2, and 2 left over toward making another bundle of 4.")

"You may write the story on the board." ($10 \div 4 = 2\frac{2}{4}$.)

$9 \div 2 =$

"You may make this." ()

(The teacher should send one and then another to the blackboard to write and "talk" the whole story until several have written it.)

("9 has how many 2's in it? It has 4, and 1 left over toward making another bundle of 2.") ($9 \div 2 = 4\frac{1}{2}$.)

$8 \div 5 =$

"You may make this." ()

"Write the whole story on your papers."

Some one is afterwards sent to the board to write it and "talk."

$7 \div 3 =$

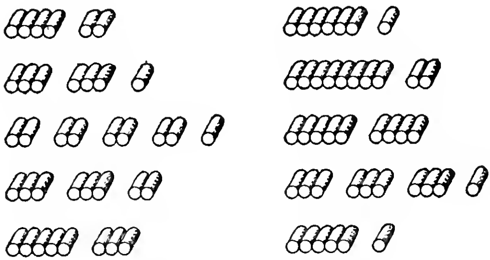
"You may make this." ()

"Write it on your papers."

(The teacher in all these exercises must see every pupil's work, objective and written.)

43. Graphic Object Reading Lessons.

Several of these exercises must be given. It is the pupil's necessary preparation for reading his seat object work when the dictation is erased.



These must be read into full expressions as follows (reading down the first column):

$6 \div 4 = 1\frac{2}{4}$
$7 \div 3 = 2\frac{1}{3}$
$9 \div 2 = 4\frac{1}{2}$ etc.

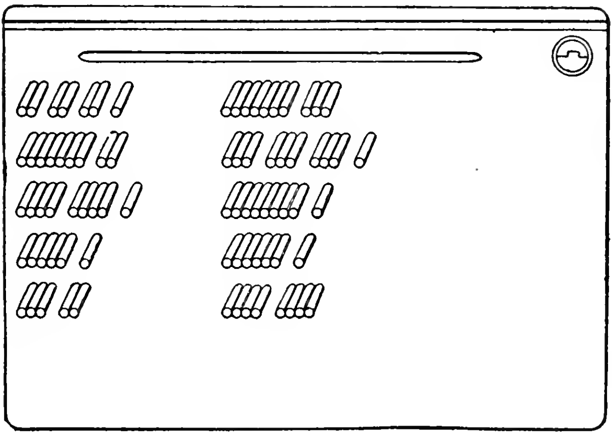
2. Seat Counting in Uneven Division.

44. These are exercises in which the child applies the written and oral language which has now been developed.

(1) Blackboard Dictation.

$7 \div 2 =$	$10 \div 7 =$
$9 \div 7 =$	$10 \div 3 =$
$9 \div 4 =$	$9 \div 8 =$
$6 \div 5 =$	$7 \div 6 =$
$5 \div 3 =$	$8 \div 4 =$

(2) Seat Construction Work.



(3) Pupil's Seat Written Work.

$7 \div 2 = 3\frac{1}{2}$	$10 \div 7 = 1\frac{3}{7}$
$9 \div 7 = 1\frac{2}{7}$	$10 \div 3 = 3\frac{1}{3}$
$9 \div 4 = 2\frac{1}{4}$	$9 \div 8 = 1\frac{1}{8}$
$6 \div 5 = 1\frac{1}{5}$	$7 \div 6 = 1\frac{1}{6}$
$5 \div 3 = 1\frac{2}{3}$	$8 \div 4 = 2$

NOTE.—After two or three weeks of this seat work, the dictations should contain an occasional expression where there are *no* whole bundles—$7 \div 8 =$, $5 \div 7 =$,

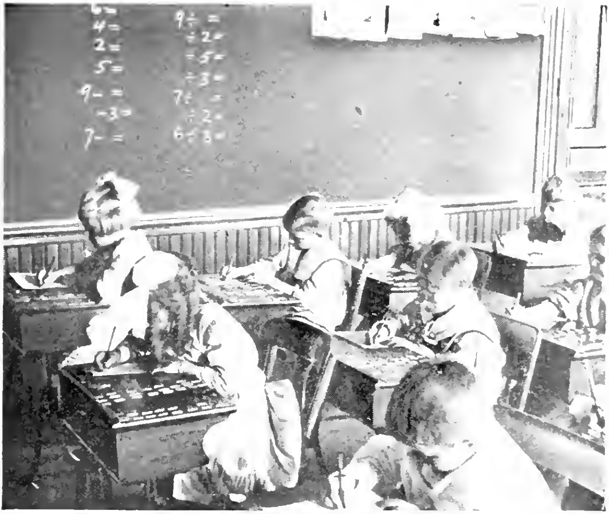
etc. $7 \div 8 = 0\frac{7}{8}$. "7 has how many 8's? It has *no* bundles and 7 toward making a bundle of 8." The development is easily made in a few minutes by a class exercise, the teacher at the blackboard. The development consists in noting that when there are no bundles the crayon or the pencil must *say* that there are no bundles. $6 \div 9 = 0\frac{6}{9}$. "Six has how many 9's in it? There are *none*, but 6 toward making a bundle of 9." Not many such expressions should be given. There should be just enough to keep the child familiar with the form of expressing them. There is little memory value in them. They need not enter into the memory tests later.

45. Uneven Division with Choice.

NOTE.—It is not necessary to outline the development plan for "choice" work in uneven division. The introduction would be similar to that for "Take Away with Choice" in Section 29. As soon as the pupils are able to do the construction work of Section 44, they are ready for work with choice. As soon as the work with choice has been learned so that the seat constructions and written work are understood, all uneven division dictations should be of the "with choice" kind. It must be continued until the memory work is complete.

Form of Blackboard Dictation.

$6 \div \quad =$	$\div 3 =$	$8 \div \quad =$
$9 \div \quad =$	$\div 5 =$	$6 \div \quad =$
$7 \div \quad =$	$\div 2 =$	$\div 4 =$
$8 \div \quad =$	$\div 4 =$	$\div 3 =$
$8 \div \quad =$	$\div 6 =$	$9 \div \quad =$



SEAT COUNTING—WITH "CHOICE."

(Section 46.)

A blackboard dictation involving "will make," "take away" with choice, and "has how many" with choice.

The pupils have completed this construction work and are writing the measurements which they have expressed objectively on their desks. In work of this kind (seat counting) it is not necessary to erase the dictation before the pupil begins written work (Section 13 (b), note).

46. Seat Counting in "Will Make" and Division—Mixed.

NOTE.—After the uneven division work is well started, the daily dictations should include "will make" work and "take away" also, provided, of course, that the "will make" work and "take away" are still in the construction stage. If the "will make" work is being done by part or all of the class without objects, those who are writing their dictations from memory should not have their uneven division and their "will make" exercises at the same hour. A dictation exercise for seat work in this or any other part of introductory number should be either all objective or all without objects.

Blackboard Dictation—Mixed.

$9 \div 5 =$	$6 \div 5 =$	$9 =$
$8 \div 7 =$	$8 \div 6 =$	$8 =$
$8 \div 3 =$	$9 \div 2 =$	$7 =$
$7 \div 4 =$	$5 \div 4 =$	$3 =$
$5 \div 2 =$	$6 \div 4 =$	$8 - \quad =$

47. Memory Tests.

What was said under "memory tests" in Section 26 is applicable to uneven division.

The form of these tests should be—

7 has how many 4's in it?
8 has how many 6's in it?
5 has how many 2's in it?

Later the questions will sometimes be in the following form:

How many 4's in 7?
How many 5's in 9?

48. Seat Work without Objects.

What is said in Section 34 about work without objects is applicable here. This work must be deferred until the oral memory work is strong, so that pupils write out their exercises rapidly without hesitation or counting. When this "seat work without objects" is thus begun, further use of objects in uneven division is not necessary.

CHAPTER X

COUNTING WITH TENS

SUMMARY:

The oral and written language (Section 50).
  (*a*) Tens terms.
  (*b*) Tens expressions
  (*c*) Seat work.
Counting (Sections 53—57).
  (*a*) In + and ×.
  (*b*) In −.
  (*c*) In even division.
  (*d*) In uneven division.
Memory work in +, ×, −, and even division (Section 58).
Construction of numbers between 10 and 100 involving units of both orders (12, 18, 36, etc.) (Section 61).

Tens' Place in the Teaching Order.

49. That work with 10's similar to that with units of the first order is properly the next step that should be taken in number work may be seen by a glance at the history of number development, if we recognize the principle that the logical is the historical order. Count-

ing as first used by the race was with the fingers as counters. The first stage was to ten because of the number of fingers. To count beyond ten a second savage was needed each time to keep the record of the 10's. If three particular fingers of the first savage were used to indicate 3, the corresponding fingers of the second savage indicated 3 10's. When later the abacus with pebbles came into use in place of finger counting, a pebble in the first space of the abacus indicated 1; a similar pebble in the second space, a 10. The same principle applied later when the apices took the place of pebbles—a 2-apex indicating 2 or 2 10's according to the space in which it was placed on the abacus. All through these finger and abacus periods, therefore, the objective character of number expression kept before the minds of our ancestors in every use of number the fact that 10's are simply units of a higher order having the same relations as those of the lower order. The three stages in this early number unfolding are thus clearly marked:

1. Counting to 10.
2. Counting 10's.
3. The expression of quantity which required both of these orders of units—12, 15, 24, etc.

The historical order of number development, therefore, suggests a teaching order as follows:

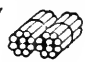
1. Constructive work with units of the first order. This has been done in Sections 1 to 48.
2. Similar constructive work with units of the second order.
3. Constructive work with combinations of these two classes of units—building such numbers as 15, 18, 35, 64, etc., to 100.

50. The Oral and Written Language of Tens.

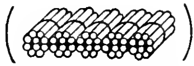
(*a*) Tens Terms.

Objects.—There should be a plentiful supply of bundles of 10's for seat work—from 50 to 100 bundles for each child. The best objects for the purpose are kindergarten inch-sticks bound with the small-size rubber bands. Shoe-pegs will answer the purpose, but it is hard for children to make the bundles with pegs. The bundles should be made by the pupils under supervision.

(1) The Oral, Concrete, and Written Language Development.

"Show me two tens." ()

"This is the way the crayon says two tens"—(20).

"Show me five tens." 

"This is the way to write 5 10's"—(50).

"Show me 3 10's." 

"Will you" (to some pupil) "write 3 10's on the blackboard?" (30.)

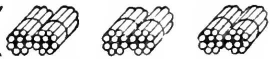
(2) Writing Exercises.

Dictate orally 4 10's, 8 10's, 6 10's, etc., for pupils to write—(40, 80, 60, etc.).

(3) Reading Exercises.

Write exercises on the blackboard (80, 60, 90, 40, etc.) for pupils to read—(8 10's, 6 10's, 9 10's, etc.).

NOTE.—The reading of 40, 20, 50, etc., as 4 10's, 2 10's, 5 10's, etc., ought to be continued as long as possible for the sake of keeping in mind through the oral language their tens nature, this being lost in large measure in the terms forty, twenty, etc. The pupils will very soon suggest the latter form and begin to use it. After this common reading (twenty, fifty, etc.) comes into general use by the pupils as their regular language in expressing such numbers, the 10's forms of expression should be preserved by occasional exercises in reading them in what may be called "the long way."

The concrete expression—()—read the long way is three two-tens (written 3 20's), or two-tens and two-tens and two-tens (written 20+20+20); read the short way it would be twenty and twenty and twenty, or three twenties.

(*b*) Tens Expressions.

The Language Development.

"Show me 2 10's." 

"Show me another 2 10's." 

"Show me another 2 10's." 

"How many 2 10's have you now?" ("Three.")

"This is the way to write it"—(3 20's).

"I will write it the other way"—(3×20).

"Show me 3 10's." 

"Show me 4 10's." 

"What have you now?" ("Three tens and four tens.")

"This is the way to write it"—(30+40).

"Show me three four-tens."

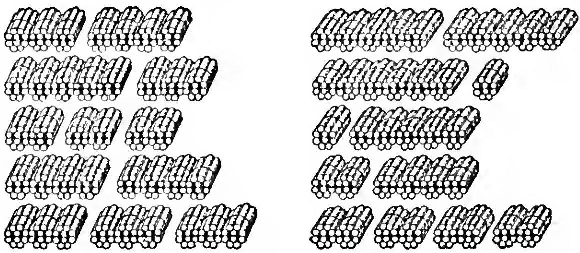
"You" (designating some pupil) "may write it on the blackboard." (3×40.)

"Show me five tens and three tens."

"You" (to some pupil) " may write it on the blackboard." (50 + 30.)

51. Graphic Object Reading.

The teacher prepares exercises on the blackboard for oral objective reading as follows:



The equal group expressions should be read in two ways in a reading exercise of this kind.

52. Written Exercises from Oral Dictations.

The teacher dictates orally, using the 10's language, of course, for pupils to write on their papers at their desks.

3×20	$80 + 90$
$40 + 20$	3×90
$30 + 40$	$60 + 40$
4×40	2×80
$50 + 50$	$80 + 60$

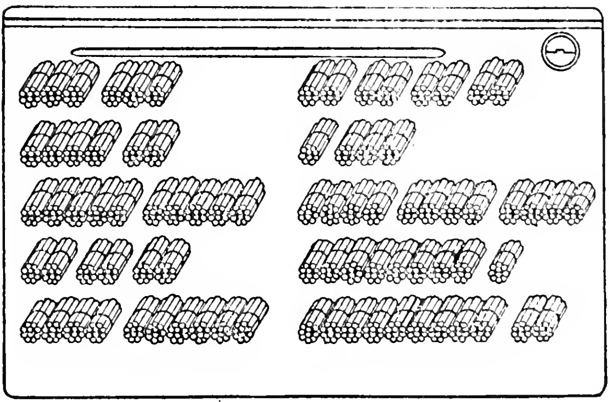
The expressions in $\times$ should be given sometimes in the 's and sometimes in the "times" language.

(c) Seat Work in Tens Language.

(1) Blackboard Dictation.

$30+30$	$20+20+20+20$
$40+20$	$10+30$
2×50	3×40
3×20	$80+10$
$40+60$	$90+20$

(2) Pupil's Seat Construction.



(3) Pupil's Written Work.

It is not necessary to give it here, as its form is apparent. The teacher must examine

the objective work as well as the written work of each pupil.

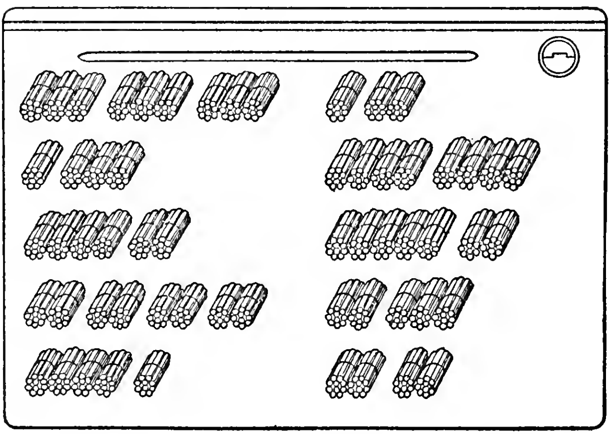
53. Tens Counting in + and ×.

This is similar to the corresponding work with units. No development work is necessary.

(1) Blackboard Dictation.

90 =	30 =
40 =	80 =
60 =	70 =
80 =	50 =
50 =	40 =

(2) Pupil's Seat Construction.



(3) Pupil's Written Work.

$90 = 3 \times 30$	$30 = 10 + 20$
$40 = 10 + 30$	$80 = 2 \times 40$
$60 = 40 + 20$	$90 = 70 + 20$
$80 = 20 + 20 + 20 + 20$	$50 = 20 + 30$
$50 = 40 + 10$	$40 = 2 \times 20$

54. If, in writing from the seat constructions, there is any tendency to use only $+$ or only $\times$ expressions, the dictations should be given, for a time at least, in the following form:

$+$	$\times$
$60 =$	$40 =$
$50 =$	$60 =$
$80 =$	$90 =$
$70 =$	$80 =$
$90 =$	$60 =$

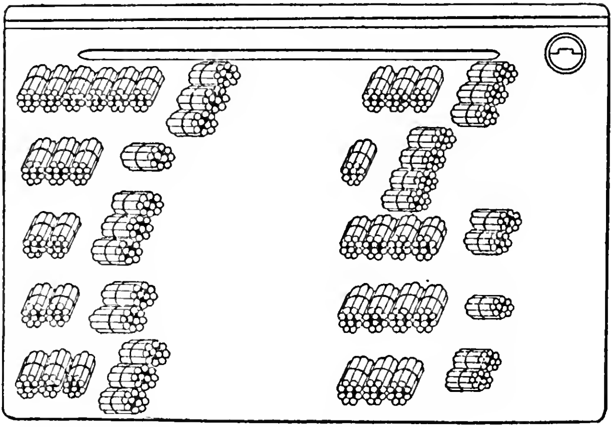
55. "Take Away" with Tens—Counting.

No development of this work will be necessary. Familiarity with the use of tens and with take away as applied to units of the first order makes it possible to give seat work at once.

(1) Blackboard Dictation.

$90 - 30 =$	$- 30 =$
$40 - 10 =$	$- 40 =$
$50 - 30 =$	$60 - \quad =$
$40 - 20 =$	$50 - \quad =$
$60 - 30 =$	$- 20 =$

(2) Pupil's Seat Construction.



(3) Pupil's Seat Written Work.

90 − 30 = 60	60 − 30 = 30
40 − 10 = 30	50 − 40 = 10
50 − 30 = 20	60 − 20 = 40
40 − 20 = 20	50 − 10 = 40
60 − 30 = 30	50 − 20 = 30

56. Memory Tests and Work without Objects.

All that is said in Sections 21, 34, and 48 about memory tests and seat work without constructions in "will make" work and "take away" with units of the first order is applicable to corresponding work with units of the second order (tens). It also applies to "even division" work in Section 57.

As soon as memory begins to be strong in

this 10's work, exercises without constructions should be given for those who can work rapidly and accurately from memory. Construction work with such pupils should not be resumed.

The 10's work does not demand the same degree of rapidity in oral or written work as does that with first-order units. The 10's work is not worth as much in mathematics.

57. Division with Tens—Even Division.

Seat construction work in *even* division should be continued until memory work is strong enough to make it wise to give exercises for seat written work without the constructions.

No introductory development is necessary.

Blackboard Dictation for Seat Work.

$90 \div 30 =$	$80 \div 40 =$
$40 \div 20 =$	$100 \div 50 =$
$60 \div 30 =$	$100 \div 20 =$
$60 \div 20 =$	$90 \div 30 =$
$80 \div 20 =$	$40 \div 20 =$

58. Uneven Division with Tens.

No attempt should be given to carry this work to the memory stage. A few days may be given to the work—enough time to enable

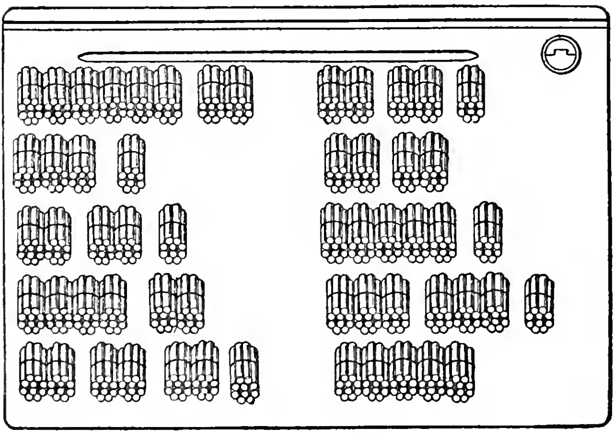
the child to learn to construct and write correctly. Work with "choice" may or may not be given as the teacher desires.

Some teachers think it wise to omit entirely this uneven division work with tens. The writer believes that it is worth doing to the extent of seat construction and writing, but not to the extent of memory work.

(1) Blackboard Dictation.

$80 \div 60 =$	$50 \div 20 =$
$40 \div 30 =$	$40 \div 20 =$
$50 \div 20 =$	$60 \div 50 =$
$60 \div 40 =$	$70 \div 30 =$
$70 \div 20 =$	$50 \div 60 =$

(2) Seat Construction.



(3) Seat Written Work.

$80 \div 60 = 1\frac{20}{60}$	$50 \div 20 = 2\frac{10}{20}$
$40 \div 30 = 1\frac{10}{30}$	$40 \div 20 = 2$
$50 \div 20 = 2\frac{10}{20}$	$60 \div 50 = 1\frac{10}{50}$
$60 \div 40 = 1\frac{20}{40}$	$70 \div 30 = 2\frac{10}{30}$
$70 \div 20 = 3\frac{10}{20}$	$50 \div 60 = 0\frac{50}{60}$

(4) The oral language of this work corresponds with that for units of the first order. $80 \div 60 = 1\frac{20}{60}$ is read, Eighty has how many sixties? It has one and twenty toward making another bundle of sixty. The other reading is, Eight tens has how many six tens in it? It has one and two tens toward making another bundle of six tens.

59. Construction and Writing to 100.

The development should be made as a class exercise—the teacher at her work-table or desk, the pupils at their seats. The teacher should supply herself with tens made of large splints, so that her constructions can be clearly seen from all parts of the room.

Development Plan.

1. The teacher constructs expressions one by one at her desk or table for pupils to read and write—20, 40, 24, 44, 53, 25, 39, etc., as follows:

 "Read it." "Write it on your papers." (20.)

 "Read this." "Write it." (40.)

 "Read this." "Write it." (24.)

 "Read this." "Write it." (44.)

2. The teacher constructs in the same way or, what is better, puts a graphic objective dictation on the blackboard for pupils to write.

Graphic Reading Exercise.

3. Blackboard Dictation for Seat Construction Work.

43	35	19
25	58	24
64	29	15
93	17	44
49	12	39

4. After the construction pupils should write from their object work, the dictation being erased.

This completes the tens work as such until it is taken up in a later grade in connection with the use of tens and units in addition and subtraction.

NOTE.—Construction Work Complete.

It should be understood that before the work of Section 60 is begun all the work already outlined must be beyond the construction stage—the stage of seat construction work. The "memory work" with units of the first order must be strong and rapid, the pupils writing the dictation work from memory (see Sections 34, 48, and 56), and that with tens must have been brought to whatever degree of strength and rapidity is desired (see Section 56).

During the time when the work outlined for tens is being done the pupils should have daily, not in co-nection with it but as a separate line of work given at a different hour, seat work without constructions with the first-order units (Sections 34 and 48). This is to keep that work fresh while working with the units of the order of tens.

CHAPTER XI

REVERSED COUNTING

60. Counting Language Reversed.

(*a*) The construction work with numbers to 10 which has now been brought to a close, its purpose to enable the child to learn the facts of these numbers through personal experience having been accomplished, required all + and × work, written or oral, to be in the analytical or "will make" form—$9=5+4$, $6=3+3$, $8=4\times 2$, etc.

(*b*) For practical work later on, we must reverse this order and accustom the pupil to the synthetic form used in ordinary language —$5+4=9$, $3+3=6$, $4\times 2=8$, etc. This outline will refer to this as "reversed language," because of its relation to the language heretofore used. It would seem strange that a child who knows instantly, when questioned, any of the groups that eight will make or seven will make should not understand what is meant by $2+6=$, $2\times 4=$, or $4+3=$, when written on the blackboard, or by "3 and 5 are how many?" or "2×4 are how many?" when given orally.

(*c*) Such, however, is the case with all

children, the brightest and the dullest. A written or an oral question of that kind has never come to them before. Their experiences—the seat construction work—have had to do with the analytical aspect only of group counting. The oral "how many" and "how much" and the written $3+2=$, and $3\times2=$, are expressions of synthetic processes with which they have had no corresponding experiences.

NOTE.—Let it be repeated here that the work outlined in this chapter must not be taken up until all construction work has been discontinued (see Section 59, note). Objects *must not* be used in the work of this section nor in any other form of "*reversed*" work.

Development Plan.

The development is very simple—the whole being begun and completed in from three to ten minutes. The lesson plan is given here to show how the change may be made by a pure development.

61. The "Reversed" Language—Development.

"I am thinking of three and two" (the teacher at the same time writing it on the blackboard without the sign of equality) —$(3+2)$.

"What did I make it from?" ("Five.")

"Yes, three and two, I made it from a five" (completing the written expression while talking—"I made it from a" being spoken while writing the equality sign).

"I am thinking of three threes" (writing it on the blackboard as before)—(3×3).

"What was it made from?" ("Nine.")

"Yes, three threes, it is from a nine" (completing and talking as before—the equality sign being written with "it is from a").

"I have $3+4$" (writing it as before on the blackboard).

"What is it from?"

"You (designating some pupil) may finish writing it for me." ($3+4=7$.)

"Yes, three and four are seven."

"I will write 2×4."

"You (to some pupil) may finish it." ($2\times 4 = 8$.)

"Yes, 2 times 4 is from 8."

62. Dictation for Seat Writing.

Exercises of this kind are dictated orally for pupils to write on paper and complete. The expressions are written one by one and not erased.

$3 \times 2 =$	$5 + 4 =$
$4 + 3 =$	$3 + 3 =$
$2 + 5 =$	$6 + 3 =$
$4 \times 2 =$	$5 + 6 =$

After the exercise has been written on the papers, some pupil should complete the blackboard dictation. To familiarize the pupils with "are" the teacher may read the completed dictation—

3 times 2 are 6.	5 and 4 are 9.
4 and 3 are 7.	3 and 3 are 6.
2 and 5 are 7.	6 and 3 are 9, etc.

The oral work may now be given in the "are" and "how many" language. The reversed form should be used exclusively in all written work.

63. Seat Work in Reversed Counting.

The dictations for daily seat work should involve all the processes, but especially that of uneven division. *Object work ceases entirely on the facts from* 2 *to* 10 before the introduction of this reversed work.

$3 + 5 =$	$9 \div 5 =$	$10 \div 7 =$
$4 + 4 =$	$6 \div 4 =$	$7 \div 4 =$
$9 - 3 =$	$5 \div 6 =$	$9 \div 4 =$
$6 - 4 =$	$5 \div 4 =$	$10 \div 6 =$
$4 \times 2 =$	$8 \div 3 =$	$9 \div 5 =$

As construction (objective) work has been discontinued (Section 60 (*c*), note), the child has the time formerly given to that to devote to written work. This makes possible much written work if desired.

This ends the work of Grade One.

CHAPTER XII

"WILL MAKE" WORK FROM TEN TO EIGHTEEN

64. The construction work should now be resumed, but *confined to the numbers above ten.* The facts of the numbers from two to ten are to be kept fresh by frequent oral tests and by occasional written exercises like those suggested in Section 63. It must be borne in mind that the written work on the facts from 2 to 10 is to be in the reversed form (Sections 60 to 63) *and without objects.* The work of this chapter (facts from 10 to 18) is in the "will make" form and *objective.*

The dictation exercises should involve all the four processes developed, but uneven division should be given the prominent place.

For the first two or three weeks "will make" exercises should be given entirely, then "will make" and "take away with choice," then exercises involving uneven division also.

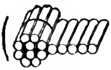
65. Range of Work.

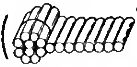
The dictation from the outset should involve all the numbers from eleven to eighteen, each exercise from the beginning having numbers along this whole range.

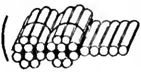
66. Development Steps.

There is no new language to develop excepting the blackboard dictation language—what to do with the bundles in dictations like 16= and b 14=, etc.

(*a*) Review, if necessary, the construction of numbers from 10 to 100 (see Section 59), but specially those from 11 to 19.

"Show me 15." ()

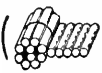
"Show me 18." ()

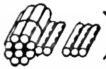
"Show me 25." ()

"Show me 11." ()

(*b*) The Use of the Tens Groups.

In this construction work the pupil should make use of the *bundles* of tens, at least for several weeks. It is not advisable even then to discontinue the use of the actual bundles.

In a construction for 15=, he should take a ten and a five () and not 15 separate objects. His construction may or may not require the breaking of this ten. If he chooses to construct 15=12+3, the ten is not broken

(); if he chooses to make $15 = 7 + 8$, the ten must be broken ( ).

(*c*) Controlling the Use of the Tens.

It will be necessary to keep the control of the breaking of the tens in the teacher's hands; otherwise there would be no way to secure experience over the whole range of the numbers 11 to 18 without interfering with the child's freedom in construction. Control, however, unless veiled takes from the work its chief claim to being a natural activity.

The control must be a part of the dictation. In no other way can it be concealed.

The following dictation suggests a method of control:

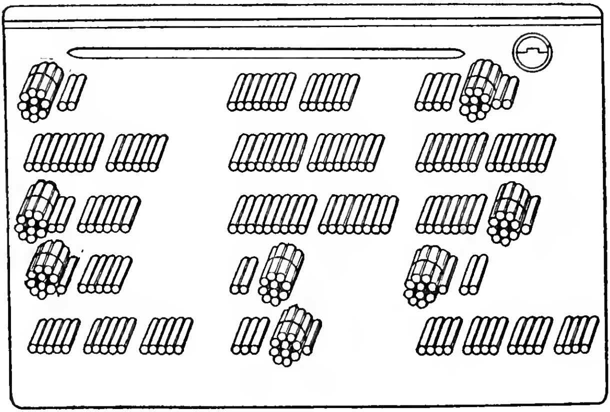
(1) Blackboard Dictation.

12 =	b. 13 =	16 =
b. 14 =	b. 16 =	b. 14 =
18 =	b. 17 =	18 =
16 =	12 =	13 =
b. 15 =	14 =	b. 16 =

The pupil is taught that the b. means that he must make something that requires the breaking of ten. From expressions which are not marked with the b., constructions are made which do not require the ten to be

broken. The following is a construction that may be made from the above dictation:

(2) Seat Construction Work.



67. Forms of Dictation Exercises.

The following exercises indicate the forms that the dictation for this work (from 11 to 18) will take. The form of the construction work, so far as anything new is involved, has been fully shown, and it will not be necessary to illustrate further.

(*a*) "Will Make" Work.

b. 12=	b. 11=	17=
14=	13=	17=
18=	15=	15=
b. 16=	b. 15=	b. 14=
16=	15=	b. 18=

(*b*) "Will Make"—Processes Separated.

+	+	×
b. 13 =	b. 15 =	18 =
b. 14 =	b. 12 =	18 =
17 =	16 =	16 =
b. 17 =	b. 16 =	14 =
18 =	b. 14 =	14 =

It is evident that it is not necessary to mark the numbers in the times column with b. The breaking can not be avoided.

(*c*) "Will Make" and "Take Away."

15 =	− 8 =	− 9 =
16 =	− 6 =	15 − =
b. 18 =	18 − =	16 − =
b. 14 =	17 − =	b. 14 =
b. 12 =	14 − =	16 =

(*d*) Uneven Division.

16 ÷ 9 =	14 ÷ 9 =	16 ÷ 9 =
18 ÷ 4 =	13 ÷ 8 =	15 ÷ 8 =
17 ÷ 8 =	18 ÷ 9 =	16 ÷ 3 =
15 ÷ 6 =	18 ÷ 7 =	14 ÷ 9 =
14 ÷ 7 =	18 ÷ 5 =	17 ÷ 5 =

(*e*) Uneven Division with Choice.

16 ÷ =	÷ 9 =	÷ 5 =
18 ÷ =	18 ÷ =	÷ 6 =
÷ 9 =	16 ÷ =	÷ 8 =
÷ 8 =	17 ÷ =	÷ 3 =
÷ 7 =	14 ÷ =	÷ 4 =

NOTE.—It is apparent that this "uneven division" counting carries with it addition, subtraction, and multiplication experiences, the pupil getting from it almost the same forms of counting as from them.

Again, the pupil finds it the most pleasurable of the construction exercises. For this reason the suggestion is repeated here that, while each of the other forms of construction work should receive proper attention, that in uneven division should be given most frequent use.

CHAPTER XIII

MEMORY AND REVERSED WORK—TEN TO EIGHTEEN

68. Memory Tests and Work without Objects.

THE suggestions for oral memory tests given in Sections 21 and 22, and those for work without objects in Sections 34 and 48 must be applied to the work on the numbers between 11 and 18. The forms of the dictations for this work without objects are similar to those under Section 67.

69. Counting Language Reversed.

Suggestions were made in Section 60 as to reversing the forms of the + and × dictations. There will be no need of repeating the development steps again in this connection, but the other suggestions in that section must be applied to this part of the work (numbers 11 to 18).

Let it be repeated here that *before the reversion begins all construction work ceases* and is not resumed. The dictations will be in the following forms, the uneven division form being given most:

$9+8=$	$4\times 3=$	$18\div 7=$	$17\div 7=$
$3\times 6=$	$4\times 4=$	$17\div 8=$	$18\div 12=$

$8+9=$	$2\times 9=$	$17\div 9=$	$16\div 11=$
$7+6=$	$17-8=$	$16\div 6=$	$14\div 9=$
$3\times 5=$	$14-9=$	$13\div 9=$	$14\div 6=$
$18-9=$	$18-6=$	$14\div 5=$	$15\div 6=$
$16-9=$	$16-7=$	$14\div 11=$	$13\div 5=$

CHAPTER XIV

NUMBER FACTS IN ADDITION AND SUBTRACTION FROM 18 TO 100

70. Form of the Work.

(*a*) The number facts in addition and subtraction between 18 and 100 are not learned and used as independent combinations. We reason them from our knowledge of the facts between 1 and 18. We know 37 and 6, 87 and 6, 67 and 6, etc., because we know 7 and 6. When we compute 84−8, 44−8, 24−8, etc., we think 14—8. No new facts are learned. The processes are simply repeated inferences from 1 to 18 experiences.

(*b*) Since, therefore, our work upon the facts of this number range (1 to 18) has been brought to the memory stage, the pupil thus having all the material and power that this work from 18 to 100 requires, this is the proper time to begin to give him the experiences that will enable him to discover these number relations for himself and to get the number facts above 18 by reasoning processes based upon them.

NOTE.—The work of this section will require many months, with short exercises almost daily. It must

run parallel with the work of the sections following, and be continued at least until the Third Grade part of this Outline is completed. It will *not* be necessary to *delay the work* of any *following section because of this work*, for the reason that no development or introductory exercise of any kind is necessary. The first and every succeeding exercise under this section is given on the blackboard without remark or suggestion. Any explanation or comment would interfere with the pupil's right to discover the relations and how to reason out the number facts. It is understood, of course, that *this work is not objective.*

71. Stages of the Work.

(1) In order that the work may progress in such a way that the pupil from the very beginning makes use of his 1 to 18 experiences unconsciously—does not hesitate for his answers or count them out on his fingers or otherwise—it must be arranged in the following order, each of the six principal stages (1 to 4, 1 to 5, 1 to 6, etc., to 1 to 9) requiring several weeks of time before the next one is taken up:

1st Stage—Dictations in which the number to be added or subtracted is 1, 2, 3, or 4.

2d Stage—Dictations using only 1, 2, 3, 4, or 5.

3d Stage—Dictations using only 1, 2, 3, 4 5, or 6, etc.

(The remaining stages are similarly outlined—1 to 7, 1 to 8, 1 to 9.)

(2) Each of these stages has two periods:

(*a*) In which the computations do not change the number decade.

(*b*) In which the number decade changes.

(3) The following blackboard dictation exercises illustrate the forms of the dictations for the several "stages:"

First Stage, Period (*a*).

$3+4=$	$6+3=$	$2+7=$
$43+4=$	$56+3=$	$2+37=$
$73+4=$	$36+3=$	$2+97=$
$93+4=$	$96+3=$	$2+67=$
$23+4=$	$86+3=$	$2+47=$

Some of the exercises in + in each stage and period should be in the form of this third column.

First Stage, Period (*a*).

$9-3=$	$8-4=$	$7-2=$
$29-3=$	$58-4=$	$67-2=$
$99-3=$	$28-4=$	$37-2=$
$59-3=$	$98-4=$	$17-2=$
$69-3=$	$78-4=$	$47-2=$

First Stage, Period (*b*).

$8+3=$	$9+4=$	$11-3=$
$18+3=$	$29+4=$	$21-3=$
$78+3=$	$89+4=$	$51-3=$
$38+3=$	$69+4=$	$81-3=$
$58+3=$	$59+4=$	$71-3=$

Second Stage, Period (*a*).

2+3=	8−5=	4+4 =
32+3=	38−5=	4+74=
72+3=	78−5=	4+24=
92+3=	98−5=	4+44=
82+3=	68−5=	4+84=

Second Stage, Period (*b*).

8+4=	9+5=	14−5=
38+4=	29+5=	94−5=
78+4=	89+5=	34−5=
88+4=	79+5=	64−5=
68+4=	49+5=	24−5=

Third Stage, Period (*a*).

2+6=	5+4=	6−3=
92+6=	25+4=	96−3=
72+6=	45+4=	56−3=
22+6=	95+4=	36−3=
42+6=	55+4=	46−3=

Third Stage, Period (*b*).

8+4=	9+6=	13−6=
68+4=	19+6=	23−6=
88+4=	49+6=	63−6=
48+4=	79+6=	43−6=
28+4=	59+6=	73−6=

It is not necessary to illustrate the other stages of the work.

Several weeks must be given to each stage before advancing to the next. It must be

borne in mind that the work of this chapter will continue with brief daily exercises for many weeks and months (through the Third or Fourth Grades) and that its inauguration and use will not delay the work of the chapters which follow.

This work must be subject to very frequent oral tests for memory. The suggestions for such tests in Sections 21 and 22, in so far as they are not inconsistent with work of this character (reversed and without objects), are applicable here.

CHAPTER XV

SHALL CONSTRUCTION WORK CEASE WITH EIGHTEEN?

72. Provision has now been made for the development of number language and for the mastery, during the process, of the number facts to 18, excepting the language and facts of partition. The limit has been fixed at 18 because at that point the "take away" work and the + work with expressions of two terms, through units of the first order, end.

Two questions now arise—Where should construction work end? Is it necessary to continue construction work until the child acquires the number facts to 9×9 or to 12×12?

This Outline has for one of its aims the development of number terms and expressions through long and consistent connection with the corresponding acts and processes of physical measurements—to cause the word or the phrase to grow into the pupil's vocabulary associated with the activity which it signifies.

Now, it is a law of the mind that when language acquired in this way is later made use of in abstract work it will give rise to mental

pictures of the essential features of the physical acts and processes around which it grew. If the child learns "take away" in connection with physical acts of subtraction and continues for a time to connect the term with the act, it is unavoidable that when he hears or reads the term in connection with his daily work it will produce in his mind an image of the act with which the term has been associated. This mental picture will have in it a whole, an act of taking away, and a group remaining—the value of the latter a matter of memory from many former experiences.

The same may be said of any other of the number terms. If "has how many" has found its way into his vocabulary in connection with physical groupings of the proper kind, the use of the term will produce a mental picture of the acts and processes of division. The image will include the whole to be grouped, the grouping process, and, in a case of uneven division, the group "left over toward making another group," etc. The number of groups will be a memory product.

Now, this power of imagery—this recognition of the effect of the process upon the quantity—is the foundation upon which success in mathematical work rests.

If continued construction work beyond 18 will add to that power, it must not be abandoned. If, on the other hand, there would be no gain, the pupil should have this part of his time for other forms of work.

Let us examine with reference to this question of imagery some number fact beyond the range of our construction work. The first step in advance from 18 would be some expression in 19, as $19 \div 8 =$. Let us consider this, keeping in mind for comparison $18 \div 8 =$, which the child who has completed this Outline to 18 knows. This problem with its solution has four parts—(1) the 19, which is a perfectly familiar number; (2) the grouping process, the nature of which the pupil knows through hundreds of experiences; (3) the number of groups (which $18 \div 8 =$ tells him); and (4) the 3 "left over". The value, 3, of the part "left over" in this case must come to him by reasoning; but, once this value is determined, the whole picture stands out as clear and real in his mind as any that he has previously formed through constructions. In the whole image there is not a strange element. Proof of the clearness of this image may be found by an cral test of the pupil. Ask him at the completion of the memory

work to 18, How many 8's in 19? and he will answer with only an instant of delay—this delay being easily and surely accounted for by his effort to determine the value of the part "left over." The readiness of his answer and the assurance with which he gives it are most convincing evidences that the imagery is not less vivid than if it had had a construction basis.

This work upon 19, therefore, loses nothing of clearness by being done abstractly under the conditions suggested. The same reasoning would apply to 20, then to 21, then to 22, etc., if each step is taken in a similar manner after the preceding step has been brought to the memory stage. We may safely conclude, therefore, that the number imagery need not suffer because of purely abstract work above 18.

CHAPTER XVI

THE FACTS ABOVE EIGHTEEN IN × AND ÷

73. Seat Work upon 19.

(*a*) This, of course, is to be without objects. It will consist of written work from blackboard dictations.

(*b*) The *introductory* work upon this step and upon each succeeding step—20, 21, 22, etc.—must be in the form of *uneven division.* The reason for this is that uneven division is the simplest form of work for the pupil after the experiences of the construction period. Following this is "take away," then "and" and "times" work "reversed."

(*c*) The time to be given to 19 before taking up 20 should be three to five days, perhaps more. The work upon 20 should not be begun until that upon 19 is in the memory stage. Each succeeding step above 19 up to 30 or 35 will require a longer time than the step preceding. The memory stage must be reached in each case before the advance is made.

(*d*) Oral memory tests must be given daily covering the whole range of number work from 4 or 5 to and including the special number

which forms the advance step upon which the pupil is working at the time.

The suggestions in Section 70, note, must not be overlooked.

(*e*) Blackboard Dictation—showing form of work for the introductory period.

$19 \div 6 =$	$14 \div 6 =$	$16 \div 4 =$
$16 \div 5 =$	$19 \div 7 =$	$18 \div 6 =$
$18 \div 9 =$	$13 \div 9 =$	$19 \div 4 =$
$19 \div 8 =$	$16 \div 7 =$	$14 \div 5 =$
$15 \div 6 =$	$19 \div 5 =$	$19 \div 3 =$

Blackboard Dictation—showing the "take away" stage.

$19 - 3 =$	$19 \div 11 =$	$19 - 6 =$
$18 - 6 =$	$19 \div 9 =$	$18 - 12 =$
$19 - 5 =$	$19 \div 12 =$	$19 - 7 =$
$19 - 4 =$	$19 \div 14 =$	$18 - 13 =$
$19 - 2 =$	$19 \div 10 =$	$19 \div 17 =$

Blackboard Dictation—"and" and "times" reversed.

$9 + 8 =$	$3 \times 5 =$	$11 + 7 =$
$14 + 4 =$	$4 \times 3 =$	$9 + 10 =$
$13 + 6 =$	$3 \times 6 =$	$19 - 7 =$
$12 + 4 =$	$18 \div 8 =$	$14 - 5 =$
$12 + 7 =$	$19 \div 7 =$	$3 \times 4 =$

74. Seat Work upon 20.

This is to be begun after the work upon 19 is in the memory stage.

Blackboard Dictation—introductory stage.

$20 \div 6 =$	$17 \div 5 =$	$15 \div 7 =$
$18 \div 5 =$	$15 \div 8 =$	$19 \div 4 =$
$19 \div 7 =$	$16 \div 9 =$	$20 \div 4 =$
$20 \div 8 =$	$20 \div 9 =$	$20 \div 7 =$
$20 \div 5 =$	$20 \div 12 =$	$20 \div 10 =$

Blackboard Dictation—"take away" stage.

$18 - 6 =$	$20 \div 8 =$	$19 - 4 =$
$20 - 4 =$	$20 - 3 =$	$18 \div 6 =$
$20 - 6 =$	$19 - 6 =$	$20 \div 7 =$
$19 - 8 =$	$20 \div 9 =$	$16 - 8 =$
$20 - 5 =$	$20 \div 7 =$	$20 - 7 =$

Blackboard Dictation—"and" and "times."

$12 + 8 =$	$4 \times 5 =$	$11 + 9 =$
$14 + 4 =$	$2 \times 10 =$	$12 + 4 =$
$15 + 5 =$	$5 \times 4 =$	$20 - 6 =$
$13 + 6 =$	$3 \times 6 =$	$19 \div 7 =$
$14 + 6 =$	$9 \times 2 =$	$15 \div 8 =$

75. The Steps above 20.

(*a*) It will not be necessary to outline the work upon the steps 21, 22, 23, etc., to 81. The suggestions in Sections 73 and 74 apply

to the remainder of the work upon the number facts excepting those of partition.

(*b*) The combinations of 13's, 14's, 15's, etc. should not be used with the numbers above 20. They are not necessary for practical work in addition, subtraction, multiplication, or division, and for that reason we may properly save the pupil the additional burden upon his memory at this age—5 to 10 years.

(*c*) The number facts to 25 should be the limit for Second Grade—the maximum limit.

CHAPTER XVII

PARTITION

76. (*a*) This is the fifth fundamental operation. Those who do not make it a separate measuring process classify it as division. It resembles division in that it is a grouping process in which the quantity to be measured is made into groups of equal value. Division and partition differ in that in the former the value of the groups is given to find the number of groups, while in the latter the number of groups is given to find the value of each.

(*b*) $12 \div 3 = 4$ (4 groups of 3 units each). This is division, and this Outline limits the term to processes of this kind.

$\frac{1}{3}$ of $12 = 4$ (4 units in each of the three groups). This is partition. Very many use also the division form ($\div$) to express this kind of grouping. They use $12 \div 3 =$ to express 12 units to be made into 3 equal groups. This Outline applies the term partition to this kind of grouping, and prohibits the use of $\div$ either in development or in applied problem work when the process is of this nature.

(*c*) From the stand-point of the child partition is the most difficult of the measuring processes to learn, and for this reason must not be undertaken until after he has acquired considerable power in the others. The difficulty in learning it is not due to the form but to the language of the grouping. The other four fundamental operations make use of language which the child at the time of entering school has already acquired in his social experiences. "And" is the most common of terms; 2's, 3's, 4's, etc., are merely plurals of his common counting words; "take away" and "has how many" when they are introduced involve no new thought—merely new applications of familiar language. On the other hand thirds, fourths, thirds of, and fourths of, do not occur in the play language of children of this age. The ideas which they convey are entirely new and must be developed. The child must therefore learn the language as well as the measuring process.

(*d*) It is advisable to postpone the subject entirely until all seat objective work in addition, subtraction, multiplication, and division ceases (Sections 68 and 69). The child will then have mastered the language and facts of these processes to 18, will have acquired

some mathematical power, and will be more mature. Ordinarily under these conditions the seat object work will begin about the middle of the second school year and it should continue into the third. Power in rapid memory work will develop during this third year.

77. Oral Language of Partition.

(*a*) The first lesson should deal with the oral and later with the written language of thirds, fourths, fifths, etc., but not with *halves*. Every child of the grade in which this work is to be done has the word *half* in his vocabulary already. If he is given four, six, eight, or ten objects, he will upon request give the teacher half of them. He will without doubt also respond correctly to the request, "Give me half of them," if he has three, five, seven, or nine objects of a kind that he can easily break. Half is a term that comes to the child in his social life outside of school. To share his good things—his candy, nuts, fruits—with his companion is a part of every child's early social development, and through this sharing habit comes the idea and word half. To attempt to develop it confuses him. Later, after the "will make" work has been com-

pleted, when the teacher takes up the work in Sections 102, 108, and 109, questions involving "half of" may be used with the others, the same as if objective work with halves had been used.

Development Plan.

78. Thirds.

OBJECTS.—Pupils at their seats, with objects—pegs, inch sticks, lentils, or other kinds—for construction purposes. The teacher at her desk, with larger objects—inch blocks, splints, or long colored sticks—so that all can easily see her constructions from where they sit.

Development Steps: The Oral Language.

(*a*) "Let us all take 12." (Teacher and pupils take 12.) "I am going to make *the thirds of* 12. Watch me and see how I make them—*the thirds of* 12." The teacher makes her groups by distribution—one in each group, then a second, then a third, etc. The pupils simply look on. "See, I have made the thirds of 12. Look at them."

NOTE.—It will not be necessary to explain the meaning of thirds or to call attention in any way to the number of groups which it requires. The pupil hears the word thirds and sees the three groups made. The construction makes the connection between *thirds* and the *three* easy.

(*b*) The teacher now gathers up her 12 objects. "Let us *all* make the thirds of 12. We will make them together." Teacher and pupils now distribute the objects into thirds.

NOTE.—The teacher in making her groups must make prominent the act of distribution. The pupils must note this form of making the groups. She must insist on the pupils making their constructions in the same manner. It is very important that for several weeks, perhaps through the development steps in partition language, the pupils be required to make their groups by distribution. It serves to make prominent in thought the number of groups and the equality of the groups in each expression.

(*c*) Exercises on this Construction when All have Completed it.

"Show me *one* of the thirds of 12." (Teach the pupils to show this by putting the hand over the other thirds.)

"Show me two of the thirds of 12." (Pupils cover with their hands the other third.)

NOTE.—In all partition objective work, whether in thirds, fifths, eighths, or any other mode of group ing, the child "hides" the groups that are not to be "looked at." When the teacher asks the class to "show" any number of the thirds or fourths or eighths constructed, the child simply "hides"—covers with the hands—those not to be "shown."

The child should also be taught to "hide" the right-hand groups. If "one of the thirds" is called for, the child should cover the *two* on the right, when "two of the thirds" are to be shown, he should cover the *one* on the right. This habit of covering from the right prepares for the form of the seat objective work later.

"Show me *all* the thirds of 12."

"How many thirds are all the thirds of 12?" ("Three.")

"Count them." (The child counts the groups—"one, two, three.")

(*d*) "Let us all take nine."

"Let us make the *thirds* of 9."

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Every one—teacher included—makes the construction.

"Show me one of the thirds of 9."

"How many do you hide?"

"Show me all the thirds of 9."

"How many thirds are all the thirds of 9?"

"Show me two of the thirds of 9."

"How many do you hide?"

(*e*) The Written Expression (Using the Construction in (*d*)).

"This is the way the crayon says one of the thirds of 9"—($\frac{1}{3}$ of 9). The teacher writes it on the blackboard, "talking" as she writes each term of it. This is the only written partition expression that need be taught. The pupils will easily infer from this how to express the others.

"You" (indicating some pupil) "may make the crayon say two of the thirds of 9"—($\frac{2}{3}$ of 9).

"You" (indicating another pupil) "may make the crayon say "all the thirds of 9"—($\frac{3}{3}$ of 9).

NOTE.—Oral "exercises on the construction" like those under (*c*) and (*d*)—(Show me one of the thirds, Show me all the thirds, How many do you hide? etc.)—and exercises in writing partition expressions as suggested in (*e*), should be given on every construction throughout this development work. In addition to this, questions on the terms of the expressions written on the blackboard to bring out the significance of these terms should be asked. The following are some of these questions applied to expressions of thirds and fourths:

(*f*) Questions on Written Work.

This section shows the questions which should be used in whole or in part in each expression as it is written on the blackboard through the whole development period to Section 90.

$\frac{2}{3}$ of 9. (Suppose this to be an expression on the blackboard.)

What says *thirds?* (The child points to the 3.)

What says *two?* (The child points to the numerator.)

What tells how many bundles to make? (The 3.)

What tells how many sticks you took? (The 9.)

How many bundles do you hide? (One.)

$\frac{1}{4}$ of 8. (Let this be used here to illustrate the kind of questions.)

What says *one?*

What says *fourth?* (The child points to the denominator.)

What tells how many bundles to make? (The child points to the denominator.)

What tells how many bundles to show?

How many bundles must we hide? (Three.)

$\frac{4}{4}$ of 12. (Let this expression be used to make further illustration.)

What does this say (pointing to the numerator)? (It says 4.)

What does this say (pointing to the denominator)? (It says 4ths.)

What does this tell me (pointing to the number)? (It tells me how many sticks to take.)

What tells how many bundles to look at? (The child points to the numerator.)

How many bundles must we hide?

What says all the 4ths of 12?

What does this (the 12) tell?

These simply indicate the various ways in which the child should be questioned.

(*g*) "Let us take 6."

"Let us make the thirds of 6."

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Questions and other exercises on the object work as in (*c*) and (*d*). The written expressions as in (*e*) and the questions suggested in (*f*). Observe suggestion as to "hiding" in (*c*) note.

(*h*) "Take 15."

"Make the thirds of 15."

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Exercises as suggested in (*g*).

79. Fourths.

Development Steps—Language.

(*a*) "Let us all take 12."
"Let us all make the fourths of 12."

No explanation will be necessary. The connection of thirds with three makes the inference of fourths from four easy.

"Show me *one* of the fourths of 12." (The child covers with the hand, as suggested in Section 78 (*c*), note.)

"Show me *two* of the fourths of 12."

"Show me all the fourths of 12."

"You" (to one of the pupils) "may write on the blackboard one of the fourths of 12." This the pupil will do without help or suggestion. It is another easy inference from his experience with thirds. ($\frac{1}{4}$ of 12.)

"You" (to another pupil) "may write two of the fourths of 12." ($\frac{2}{4}$ of 12.)

"You may write all the fourths of 12." ($\frac{4}{4}$ of 12.)

With the object work, make use of the exercises suggested in Section 78 (*c*) and (*d*).

With the written expressions, question as in (*e*) and (*f*) of the same section.

(*b*) "Take 16 sticks."

"Make the fourths of 16."

(*c*) "Take 8 sticks."

"Make the fourths of 8."

(*d*) "Take 20 sticks."

"Make the fourths of 20."

(*e*) Give a few more exercises of the same kind.

NOTE.—With each of these exercises the suggestions in Section 78 (*c*), (*d*), (*e*), and (*f*), must be followed.

80. Thirds and Fourths.

(*a*) "Take 6."

"Make the thirds of 6."

"Show me *two* of the thirds of 6."

"Show me *all* the thirds of 6."

"Show me *one* of the thirds of 6."

"Write *one* of the thirds of 6." ($\frac{1}{3}$ of 6.)

"Write *all* the thirds of 6." ($\frac{3}{3}$ of 6.)

"Write *two* of the thirds of 6." ($\frac{2}{3}$ of 6.)

Note.—The suggestions as to exercises on the objective, and the questions on the written work must be observed in this and in each of the following exercises.

(*b*) "Take 12."
"Make the fourths of 12."
"Show me *three* of the fourths of 12."
"Show me *all* the fourths of 12."
"Write *one* of the fourths of 12."
"Write *three* of the fourths of 12."
"Write *all* the fourths of 12."

(*c*) "Take 8."
"Make the fourths of 8."
Have the pupils show and write each of the forms suggested in (*a*) and (*b*).

(*d*) "Take 12."
"Make the thirds of 12."
Follow the plan outlined in (*a*).

81. Sixths.

(*a*) "Take 12 sticks."
"Make the sixths of 12."
"Show me *four* of the sixths of 12."
"Show me *one* of the sixths of 12."
"Show me *all* the sixths of 12."
Have the pupils show the other sixths of 12 in the same manner.
"Write *one* sixth of 12."

It will not be necessary to teach the child how to write this.

"Write *five* sixths of 12."

Have all the forms written. (The oral exercises and questions as before.)

(*b*) "Take 18."

"Make the sixths of 18."

Have the child "show" and write the several forms. Give the oral exercises.

(*c*) One or two additional exercises may be given.

82. Fifths.

(*a*) "Take 15."

"Make the fifths of 15."

"Show me *one* of the fifths of 15."

Have the child "show" and write the forms as before and give the oral exercises.

(*b*) "Take 20."

"Make the fifths of 20," etc.

(*c*) "Take 10."

"Make the fifths of 10," etc.

83. Sevenths, eighths, and ninths may now be taken in the same manner, the suggestions as to the oral work being followed with each expression.

Miscellaneous exercises with thirds, sixths, fourths, eighths, etc., not beyond twelfths, should now be given.

NOTE.—Use any desired numbers in connection with this work, but it is not very profitable to go beyond 30 at this time on account of the time required to count the sticks. All this work should be conducted with energy, so as to hold the attention that follows rapidity of action.

84. Exercises to Emphasize the Equality of Partition Groups.

Use long splints that can be easily broken into equal parts.

(*a*) "Take 7."

"Show me the thirds of 7."

NOTE.—There will be troubled looks when the pupil after distributing 6 finds that there is an odd stick. He will doubtless, after some consideration, drop this object either back into the box with the unused objects or into one of the groups. To make a pure development, the teacher's work is simply to call the child's attention in the one instance to the fact that if he puts the object back into the box he will have used but 6, and consequently will not have the thirds of 7, in the other instance to the fact that his bundles are not all alike. This must be repeated, if necessary, again and again. The work must be at a stand-still. The child must be given time to think. The teacher should, if possible, avoid further suggestion in the matter. It may be necessary finally, however, to suggest that the stick might be broken, but

usually not. Some pupil will think of it and break the stick. Others will follow and the constructions will be completed.

After the pupil gets the idea that the stick may be broken to complete the groups, the teacher must watch his work to see that he attempts, at least, to break it into equal parts. If a child's pieces are not equal, they should be thrown away and he should be given another stick in place of it, to try again. To measure the stick carefully with the eye in order to get the parts as nearly equal as possible is a very essential feature of the child's work in exercises of this kind. It is not necessary that these parts should be exactly equal, but it must be apparent that the child has *tried* to make them equal. We want the thought and the effort. The oral exercises on the objective work suggested in (*c*) and (*d*) and the "questions on written work" in (*f*) of Section 78 must be followed here, but the pupil *must not be asked how many objects in each bundle.* This question *must not* come up.

(*b*) "Take 9."
"Show me the fourths of 9."

(*c*) "Take 13."
"Show me the sixths of 13."

(*d*) "Take 11."
"Show me the fifths of 11."

(*e*) "Take 16."
"Show me the sevenths of 16."

(*f*) Use several other similar exercises.

Note.—In each of these exercises—(*a*), (*b*), (*c*), (*d*), and (*e*)—follow the suggestions as to questions on the objective work and on the written expressions.

The purpose of this work is simply to emphasize the fact of the equality of the groups. With the exercises outlined above, this work of uneven partition ceases. It is not carried further and *is not* to be *repeated later. It stops here.*

85. Reading Partition Constructions, Partial.

The teacher constructs partition expressions for the pupils to interpret orally and write. The work is done at the desk or work-table with inch blocks, splints, or other large objects that can be seen by the pupils from their desks. This is called partial reading because the teacher tells the pupil the number of objects "taken."

(1) "I will take 12."

"I will make something and will see if you can tell what it is." (The teacher now makes the fourths of 12 by distribution.)

"What did I make?" ("The fourths of 12.")

"What is this?" (covering the three right-hand fourths). ("One fourth of 12.")

"Write it."

The questions suggested in (*f*) of Section 78 must be used in all these exercises.

"What is this?" (covering the right-hand fourth).

"Write it."

"What is this?" (all uncovered). ("It is four fourths of 12.")

"Write it."

"What is this?" (covering two fourths).

"Write it."

(2) "I will take 14 and make something." The teacher makes sevenths by distribution.

"What did I make?" ("You made the sevenths of 14.")

"What is this?" (covering three groups). ("It is four sevenths of 14.")

"Write it." (Use the questioning suggested in (*f*) of Section 78.)

"What is this?" (covering one group). ("It is six sevenths of 14.")

"Write it."

Have other exercises by covering different numbers of groups and writing the expressions.

(3) Have a similar exercise with 15, making fifths.

(4) Have a similar exercise with 18, making sixths.

(5) It may be necessary or desirable to give additional exercises of the same kind.

86. Reading Partition Constructions, Full.

The teacher takes any desired number of objects without telling the pupils how many. In Section 85 the teacher stated the quantity to be measured in each case—12, 14, 15, 18, etc., the pupil discovering from the teacher's work the kind of grouping—thirds, fourths, or sixths, etc. In this section the pupil must discover while the teacher is constructing the expression both the quantity being measured and the kind of grouping. With this exception the work is the same and follows the same plan as suggested in that section.

(1) "What have I made?" (The teacher has taken 6 and made thirds, by distribution.) ("You have made the thirds of 6.")

"What is this?" (covering one group). "Write it." (The "oral exercises" on the written work in all cases.)

(2) "What have I made this time?" (The teacher has taken 8 and made fourths.) ("You have made the fourths of 8.")

"What is this?" (covering one group). "Write it."

Give other exercises on this construction—(1) covering 3 groups, (2) covering 2 groups, (3) leaving all the groups uncovered. Write and discuss each.

(3) An exercise with 10, making fifths.

"What have I made this time?"

"What is this?" (covering 2 groups).

"Write it."

Exercises, with discussion of the written expression in each—(1) covering one group, (2) covering 4 groups, (3) leaving all the groups uncovered, etc.

(4) A similar exercise with 9, in thirds.

(5) A similar exercise with 12, in sixths.

(6) A similar exercise with 12, in fourths.

(7) An exercise with 15, in fifths.

(8) An exercise with 15, in thirds.

87. Pupils Construct from Written Dictations.

In this section the teacher dictates partition expressions one by one on the blackboard. The pupil must without help construct the concrete expressions from the written. If proper attention has been given to the "oral exercises" on the written work, the child should now be able to read the expressions and know how many sticks to take, how many

groups to make, and how many groups to show.

The following exercises show the form of the work:

(1) $\frac{2}{3}$ of 6. (This the teacher writes on the blackboard.)

"Make this for me." (The pupil constructs $\frac{3}{3}$ of 6 and covers with his hand the group on his right.)

$\frac{1}{3}$ of 6. (Written on the blackboard.)

"Show me this with what you have on your desks." (The pupil simply covers two groups.)

$\frac{3}{3}$ of 6. (On blackboard as before.)

"Show me this with what you have."

With each of these exercises ask—How many groups do you hide? How many groups do you show? How many sticks did you take?

NOTE.—Pupils need not be required hereafter to make their partition groups by distribution.

(2) $\frac{5}{6}$ of 12. (Written on blackboard as before.)

"Make this for me."

$\frac{6}{6}$ of 12. (Written.)

"Show me this with what is on your desk."

$\frac{1}{6}$ of 12. (Written.)
"Show me this."
$\frac{3}{6}$ of 12.
"Show me this."
With each of the above exercises ask the questions suggested in (1).

(3) $\frac{4}{5}$ of 15.
"Make this for me."
In connection with this have the pupils show $\frac{1}{5}$ of 15, $\frac{5}{5}$ of 15, etc., as in (2). Ask questions with each.

(4) Continue this work with several other dictations, using fourths, sevenths, fifths, etc.,—not using numbers above 25 or 30.

88. Constructing from Oral Dictations.

The work of this section is like that of Section 87 excepting that the dictations are entirely oral. The same lines of work should be done and the same questions asked. This oral work naturally follows the written—constructions from oral dictations being much more difficult than those from dictations which the child gets through the eye.

(1) "Make $\frac{3}{4}$ of 8." (Oral dictation.)

NOTE.—The act of covering up the $\frac{1}{4}$ is a part of the construction. The child constructs the $\frac{4}{4}$ of 8 and then sits with his hand over this "hidden" group.

Questions on the Object Work.

"When you make $\frac{3}{4}$ of 8—
"How many sticks do you take?"
"How many groups do you make?"
"How many groups do you hide?"
"How many groups do you show?"

Questions of this kind are asked *after* the construction, not *before*.

On the same construction ask for $\frac{1}{4}$ of 8, $\frac{4}{4}$ of 8, $\frac{2}{4}$ of 8. Question as before on each.

(2) "Make $\frac{4}{5}$ of 10."
Ask also for $\frac{1}{5}$ of 10, $\frac{5}{5}$ of 10, $\frac{3}{5}$ of 10, etc.
(3) "Make $\frac{7}{7}$ of 14."
(4) "Make $\frac{8}{9}$ of 18."
(5) "Make $\frac{3}{5}$ of 20."
Give additional exercises if necessary.
Question on (2), (3), (4), etc., as in (1).

89. Form of Partition Seat Work Developed. Written Dictations.

The work of this section differs from that in Section 87 only in that the successive dictations used in the day's exercises are preserved so that they accumulate into column form. The pupil's constructions are also preserved and form corresponding columns.

NOTE.—It is here that the teacher gives the pupils an arbitrary form for showing hidden groups. When the child comes to have a second and then a third construction on his desk he can not "hide" with the hand, but must indicate what should be "covered up" in some other way. The form suggested for indicating these groups is by laying them horizontally. $\frac{3}{4}$ of 8 would be shown

it being understood that the horizontal group is "hidden"—not to be looked at. $\frac{2}{5}$ of 15 would be shown as

This section, then, has two aims—

(*a*) To accumulate dictations and corresponding constructions into column form as they will appear in the seat work later.

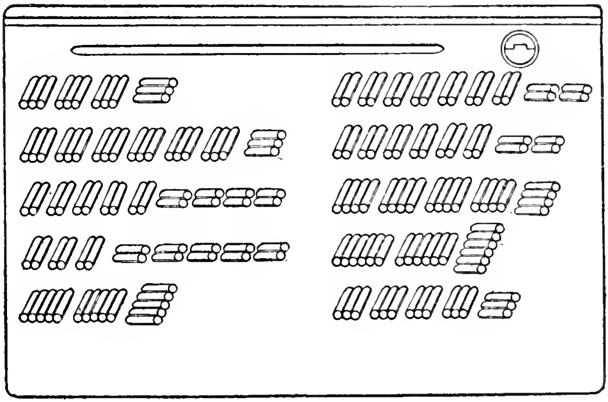
(*b*) To teach the child how he may "show" "hidden" groups without covering them with the hand.

The following illustrates a series of dictations. They are given one by one on the blackboard. The teacher writes an expression and the pupils construct, then another dictation and construction, then another, and so on—the columns of constructed work on the desk corresponding to the columns of the dictation work.

$\frac{3}{4}$ of 12
$\frac{6}{7}$ of 21
$\frac{5}{9}$ of 18
$\frac{3}{8}$ of 16
$\frac{2}{3}$ of 15

$\frac{7}{9}$ of 18
$\frac{6}{8}$ of 16
$\frac{4}{5}$ of 20
$\frac{2}{3}$ of 18
$\frac{4}{5}$ of 15

The Pupil's Seat Work.

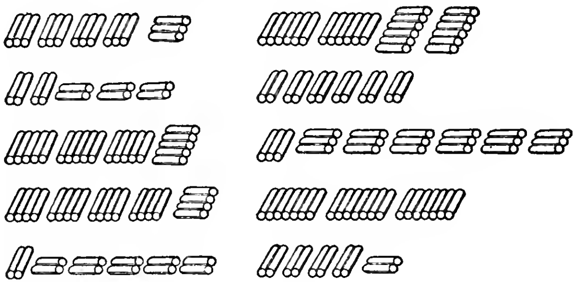


90. Reading Lessons in Objective Work.

Before seat work in partition can be begun, the child must be given exercises in reading object work. Graphic objective exercises may be used for this purpose. Several exercises should be given for oral reading—one pupil after another being called upon to read two or three expressions—and many more for written reading.

When pupils are able to read this graphic work readily, it is safe to assume that they can read their seat objective work. The graphic reading exercises may then be discontinued. In the graphic written reading the pupils all have paper and pencil and write each expression as the teacher points to it on the blackboard.

Graphic Reading Lesson.

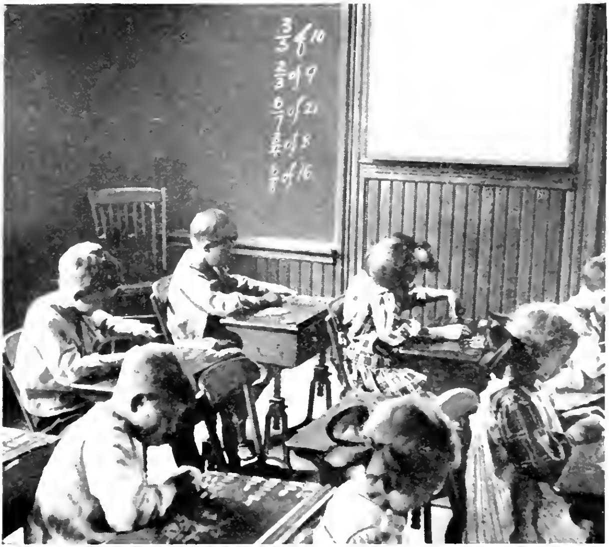


The written work from a written graphic reading lesson should be carefully inspected to see that it is correct.

NOTE.—The partition work thus far outlined will not require to exceed one month.

91. Seat Work in Partition Language.

The pupils are now ready for regular seat work. The teacher writes an exercise of proper length on the blackboard, the pupils



PARTITION LANGUAGE.

(Section 91.)

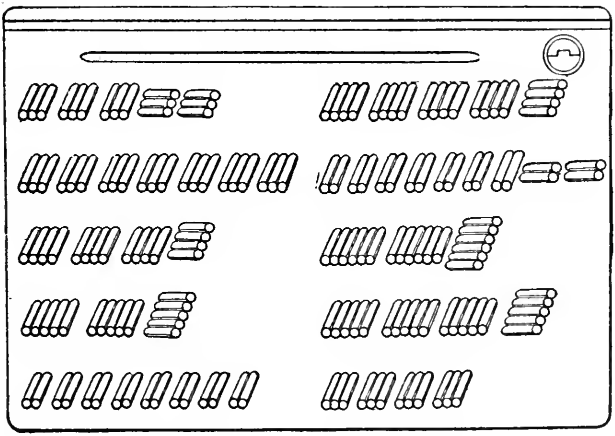
A short blackboard dictation in partition language, showing pupils at work putting the expressions into concrete language.

construct, and afterwards, the dictation being erased, write from their seat work. The work corresponds to that in Section 8. The child's objective work should be inspected before he begins to write from it. Afterwards the written work must be inspected.

(*a*) Form of Blackboard Dictation.

$\frac{3}{5}$ of 15	$\frac{4}{5}$ of 20
$\frac{7}{7}$ of 21	$\frac{7}{9}$ of 18
$\frac{3}{4}$ of 16	$\frac{2}{3}$ of 18
$\frac{2}{3}$ of 15	$\frac{3}{4}$ of 20
$\frac{8}{8}$ of 16	$\frac{4}{4}$ of 12

(*b*) Pupil's Seat Work.



This work should continue for several weeks.

NOTE.—The partition work so far has been in language development purely—the sign of equality not having been introduced. The outline has not suggested even indirect attention to the number of objects in a group. It is desirable, however, in the work from Section 82 to 91 to ask *occasionally* when there is object work on the child's desk, How many sticks are there in one of these bundles? not with a view to have the pupil remember the number but to get the observation. This suggestion must be understood not to apply to the work in Section 84. The child should have mixed objects for his seat work—pegs, inch sticks, etc.—so that he will be compelled to choose for each expression objects of one kind.

92. Oral Exercises for Language and Imagery.

This work consists of exercises *without objects* on expressions dictated orally.

(1) "$\frac{2}{3}$ of 6." (This is given orally. Pupils do *not* construct.)

"If you should make this, how many groups would be seen?

"How many would be hidden?"

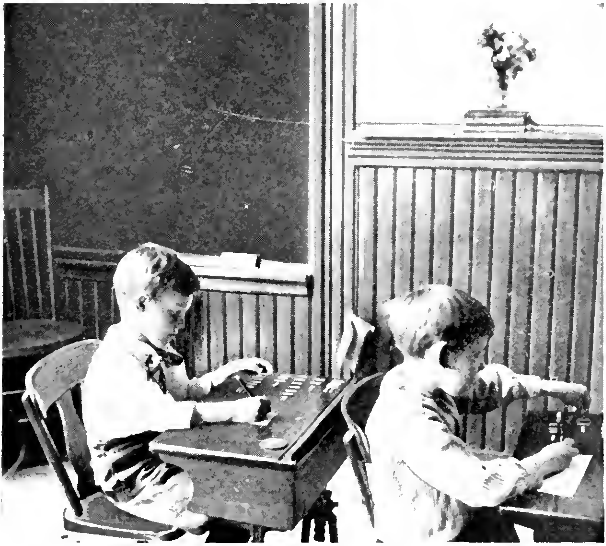
"How many sticks would you take?"

"How many groups would you make?"

"How many groups do I speak of?"

"How many are there that I do not speak of?"

NOTE.—The first and fifth questions are the most important.



PARTITION LANGUAGE.

The dictation erased and the pupil writing what the concrete work says (Section 91).

(2) Similar exercises on other expressions—
$\frac{3}{4}$ of 12, $\frac{2}{5}$ of 20, $\frac{3}{3}$ of 15, $\frac{7}{8}$ of 16, etc.

Note.—The work outlined in this section should run parallel with that in Section 91. Three or four minutes should be given to it every day if possible. Its aims are (1) to develop partition imagery by exercises which call it into use and (2) to cause the child's attention to return again and again to the fact that a partition expression, oral or written, relates to "*seen*" groups only—the "*hidden*" groups not being spoken of.

93. Partial Constructions for Language and Imagery.

This section is similar in its aims to Section 92.

The form of the work differs from that in Section 92 in that—

(*a*) The dictations are at first oral, then written.

(*b*) Objects are used.

The work here outlined should be done during the last part of the period given to Section 91.

(1) "$\frac{2}{5}$ of 10." (Oral dictation.)
"Make the *hidden* groups."

"Is the expression right?" ("No, the seen groups have not been made.")

"You may finish it." 

(2) "$\frac{3}{7}$ of 21." (Oral dictation.)

"Make the groups that we 'show'."

"Is the object work all done?" ("No, the 'hidden' groups have not been made.")

"You may finish it."

(3) Give other similar exercises from oral dictation, calling sometimes for the "shown," at other times for the "hidden" groups—$\frac{4}{9}$ of 27, $\frac{6}{7}$ of 14, $\frac{3}{3}$ of 12, etc. The value of the work will depend largely upon the care in questioning. (See note.)

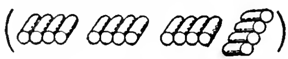
(4) $\frac{3}{4}$ of 16. (Written dictation.)

"Make the groups that we 'show'."

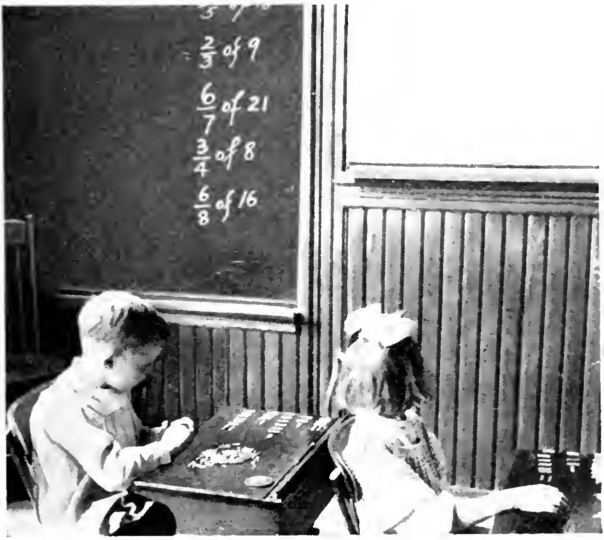
"Is the work right now?"

"What more must be done?"

"You may finish it."



(5) Other similar exercises from written dictation following the suggestions in (3) —$\frac{6}{7}$ of 21, $\frac{8}{9}$ of 18, $\frac{4}{4}$ of 12, $\frac{3}{7}$ of 21, etc.



PARTITION LANGUAGE.

The construction work completed ready for inspection (Section 19).

NOTE.—The following questions are suggested for use in this section, a part or all of which should be used with each exercise:

Under $\frac{2}{5}$ of 10. (Oral.)

In $\frac{2}{5}$ of 10 how many groups do we speak of?

How many groups do we show?

Does the expression say anything about the hidden groups? No.

The reverse form of question should be used frequently.

Under $\frac{2}{3}$ of 15. (Oral.)

What does the 2 tell? (How many groups are shown?)

What does the word thirds tell? (How many groups in all?)

What does the 15 tell? (How many sticks in all?)

Written dictations admit of another form also—

Under $\frac{4}{7}$ of 21. (Written.)

What does this tell (pointing to the numerator)? (It tells how many groups we show.)

What does this tell (pointing to the denominator)?

What does this tell (pointing to the 21)?

Questions on a written dictation after a partial construction but before the expression has been finished objectively—

Under $\frac{5}{8}$ of 16. (If the teacher has called for the hidden groups.)

Have you this part of it (pointing to the 5)? (No).

Have you this part of it (pointing to the 8)? (No, I have only the hidden groups.)

Have you this (pointing to 16)? (No, I have only a part of them.)

Under $\frac{4}{7}$ of 28. (If the teacher has called for the groups that we show.)
Have you this part of it (the 4)? (Yes.)
Have you this part of it (the 7)? (No, only a part of them.)
Have you this (the 28)? (No, only a part of them.)

94. Partial Constructions by the Teacher for Pupils to Interpret.

The aim is to give further attention to partition language by requiring the child to pass judgment on another's partial constructions.

A partition expression is stated orally by the teacher, who then makes a partial construction. The pupil states what part is made, tells what is necessary to complete the expression, and reads or writes the full statement when finished.

The work may be done in the form of class exercises, the teacher at her work-table with large objects; or as table work, the pupils in groups at the table.

NOTE 1.—In this and the following sections the teacher, in the partial constructions of partition expressions, must make either *all* the *seen* or *all* the *hidden* groups.

(1) The teacher picks up some objects the number of which is not given to the

pupils, and begins a construction, at the same time stating to the class what she is going to do—" I am going to make $\frac{3}{5}$ of something, and I want to see who can tell me when it is right and then what the full expression is."

She makes—()

Is this right? The pupils will doubtless respond that she has made only the groups that are to be "seen." She asks what remains to be done, and at the dictation of the class completes it—()

"Can you tell me now what my full expression is?" The pupils state it and write it—$\frac{3}{5}$ of 10.

(2)" I will make $\frac{5}{7}$ of something."

()

"Is this right?" ("No, you have only the hidden groups.")

At pupils' dictation the teacher completes it—()

The pupils will state and write it—($\frac{5}{7}$ of 14).

(3) "I will make $\frac{4}{4}$ of something."

"Is this right?" ("Yes.")

Pupils will state and write—($\frac{4}{4}$ of 12).

(4) "I will make $\frac{4}{6}$ of something."

"Is this right?" ("No, it is the hidden groups only.")

Pupils dictate, teacher completes—

Pupils will state and write—($\frac{4}{6}$ of 18).

(5) "I will make $\frac{5}{9}$ of something."

"Is this right?"

Pupils dictate, teacher completes—

Pupils will state and write—($\frac{5}{9}$ of 18).

Several additional exercises of the same kind should be given.

When the pupils have written the full expression in each of these exercises they should be questioned, as suggested in note in Section 93, before passing to the next exercise.

The following special questions are suggested—

> Did I say that I would make all the groups?
>
> Did I say that I would make all the "seen" groups?
>
> Did I do so?

NOTE 2.—Care must be exercised in "taking" the objects for partial constructions. The number of objects that the teacher "takes up" in each of the exercises (1), (2), (3), etc., of this section should be just enough to make the partial construction—the "seen" or the "hidden" groups—which she has in mind. In (1) she takes 6 only; in (2), when "hidden" groups are to be made, she takes the 4 only; in (5) she takes 10; etc.

When a construction is to be completed, the necessary objects for the remaining groups must be obtained from the general supply on the table or desk where the work is being done. This suggestion applies also to the work in Section 95.

This is a very important feature of this development work, as leading up to full "will make" work in Section 99. Take for illustration the expression 14 will make $\frac{7}{8}$ of 16 ($14 = \frac{7}{8}$ of 16): To construct it we must "take" 14. These objects make the "seen" groups only of the concrete expression. The "hidden" group ($\frac{1}{8}$ of 16) is not a part of the quantity that is being measured. If we wish this "hidden" group to appear in the concrete expression, we must construct it from quantity other than that "taken" —we must get the objects from our general supply.

This is not a matter that can be explained to the

child, but by a proper development process he may learn to recognize the fact.

One of the aims of Sections 94–98 is to develop this fact. Section 94 uses the objects "taken" to make at one time the "seen" at another time the "hidden" groups. Sections 96 and 98 use these objects for the "seen" groups only. The number "taken" in each of the exercises is used for a partial construction, the completing groups being made of objects not included with these. The development process consists in thus keeping the fact constantly before the child as a part of every construction experience. Hence the importance of "taking" in each case just enough objects for the partial construction contemplated.

95. Partial Constructions by the Teacher, with Emphasis on the "Seen" Groups, Construction Work Concealed.

The work of this section is similar to that in Section 94, excepting that the teacher in her partial constructions makes the *seen* groups only and the construction work is done in every case behind a book or other screen where the pupils can not see it.

The teacher works at the table with large objects that all can see; the pupils are at their desks.

The aim of the work is the same as that of Section 94. In this section attention is concentrated wholly upon the *seen* groups—those of which the expressions specifically speak.

(1) The teacher selects in her mind a certain expression—perhaps $\frac{3}{4}$ of 12.

She constructs the *seen* groups—

()—

on her desk behind a box, book, or other object so that the construction can not be seen by the class.

"I have behind this book $\frac{3}{4}$ of something."

"How many groups ought I to have in all?"

"How many *seen* groups ought I to have?"

"How many hidden groups?"

Taking away the screen—"Is the work all right?" The pupils will say at once that the *hidden* groups are wanting.

The teacher should call a pupil to the desk to complete the construction. A pupil will then state the full expression and write it on the blackboard—($\frac{3}{4}$ of 12).

Questions that may be asked after the construction is complete and the expression written—

Did I say that I had *all* the groups? (No.)

Did I say that I had all the *seen* groups? (Yes.)

Was what I *said* right? (Yes.)

Was my *construction* right? (No, it did not have the groups that we hide.)

(2) $\frac{5}{9}$ of 18. (This is supposed to be the teacher's next selection.)

Construction behind the screen—

"I have here $\frac{5}{9}$ of something."

"How many groups *ought* I to have in all?" ("Nine.")

"How many seen groups should I have?"

"How many hidden groups?"

The screen being removed—"Is the work all right?" ("No, only the groups that we *see* are there.")

A pupil is called to complete the construction.

A pupil is now selected to state and write the full expression on the blackboard—($\frac{5}{9}$ of 18).

Questions on the expression as in (1).

(3) Many more such exercises should be given. If the expression selected contains all the groups,—$\frac{4}{4}$ of 16, $\frac{3}{3}$

of 21, $\frac{6}{6}$ of 24, etc.,— when the screen is removed the pupil will find the construction complete ready for the expression to be stated and written. The work will then be "all right" as first made.

(4) Some constructions behind the screen for expressions like $\frac{2}{3}$ of 12, $\frac{4}{5}$ of 25, $\frac{5}{7}$ of 21, etc., should be made with the hidden as well as the seen groups—the complete constructions. In this way the child can not always anticipate what is behind the screen, and at the same time it will not in any way emphasize the hidden groups.

96. The First Step in Partition Measuring.

The plan of the work is similar to that in Section 95 except that the objects behind the screen are not grouped. The pupil must group them, complete the concrete work, and state and write the expression.

The teacher is at her table, with large objects and something for a screen, as in Section 95, the pupils at their desks. The screen is used for the first eight or ten exercises, after that it should be discontinued—all the work being in full view.

The teaching aim is to develop the fact that the quantity to be measured forms "shown" groups only, but *all* of these. If I say 12 will make $\frac{6}{7}$ of 14, the 12 forms the six groups and only these.

(1) The teacher puts a certain number of objects behind the screen, at the same time stating the number and what they will make.

 (These represent the objects placed behind the screen.)

"I have 8 objects here behind this book."

"They will make $\frac{2}{3}$ of something."

"Did I say that they would make all the 3ds?" ("No.")

"Did I say that they would make the *seen* 3ds?" ("Yes.")

"The groups have not yet been made and I want some one to come and make them."

The teacher now removes the screen, but before she calls the child to begin the construction she questions again—

"What did I say they would make?" ("$\frac{2}{3}$ of something.")

"Did I say they would make *all* the 3ds?" ("No.")

"Did I say they would make the *seen* 3ds?" ("Yes.")

"You may make the groups" (calling some one). (The child will make—

)

"Is the work all right?" "Does it say 3ds?" ("No, the hidden groups are not there.")

"You may finish it" (calling some one).

()

"You may give the whole expression." ($\frac{2}{3}$ of 12.)

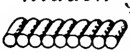
"Now we will make the crayon tell the whole story"—($8 = \frac{2}{3}$ of 12).

The teacher reads it, pointing to its sign as she speaks each word, the = being read "will make." "Eight will make two-thirds of twelve."

"When we say 8 will make $\frac{2}{3}$ of 12, do we say that it will make *all* the 3ds?" ("No.")

"Do we say that 8 will make all the *seen* 3ds?" ("Yes.")

"Do we say that the 8 will make the *hidden* 3ds?" ("No.")

(2) () Representing objects placed behind the screen.

"I have 9 objects here."

"They will make $\frac{3}{5}$ of something."

"Did I say that they would make all the 5ths?" ("No.")

"Did I say that they would make the *seen* 5ths?" ("Yes.")

"Did I say that they would make the *hidden* 5ths?" ("No.")

"The groups have not been made and I am going to call one of you to make them."

The teacher removes the screen.

"What did I say that 9 would make?" ("You said that it would make $\frac{3}{5}$ of something.")

"Did I say that it would make all the 5ths?" ("No.")

"Did I say that it would make the *seen* 5ths?" ("Yes.")

You may make the groups (calling a pupil). (The child makes—

)

"Is the expression all finished?" ("No, the hidden groups have not been made.")

"You may finish it"—

()

"Read the expression now"—($\frac{3}{5}$ of 15).

"Now we will make the crayon tell the whole story"—($9 = \frac{3}{5}$ of 15).

"I will read it"—"9 will make $\frac{3}{5}$ of 15."

"When I say that 9 will make $\frac{3}{5}$ of 15, do I say that it will make *all* the 5ths?"

"Do I say that it will make the hidden 5ths?"

"Do I say that it will make all the shown 5ths?"

(3) () (This represents objects behind the screen.)

"I have 12 objects."

"They will make $\frac{4}{7}$ of something."

"Will they make *all* the 7ths?"

"Will they make the hidden 7ths?"

The screen is removed.

"I want to ask some one to make the groups."

"Will the 12 objects make all the 7ths?"

"Will they make the shown groups?"

"You" (calling a child) "may make the groups." (The child makes—

)

"Is the expression all finished now?"

"You may finish it."

()

"What does the expression say?" ($\frac{4}{7}$ of 21.)

"We will make the crayon tell the whole story"—($12 = \frac{4}{7}$ of 21).

"You may read it." ("12 will make $\frac{4}{7}$ of 21.")

(4) () (This represents the objects behind the screen.)

"I have 10 objects here."

"They will make $\frac{2}{3}$ of something."

"Will they make all the 3ds?"

The screen is now removed.

"What did I say they would make?" ($\frac{2}{3}$ of something.)

"You" (calling a child) "may make the groups."

The child makes— ()

"Is the expression all finished?" ("No.")

"You may finish it"—

()

"What does the expression say?" ($\frac{2}{3}$ of 15.)

"I want some one to write the whole story about what 10 will make." (10 = $\frac{2}{3}$ of 15.)

"You may read it." ("10 will make $\frac{2}{3}$ of 15.")

(5)  "I have 6 objects here."

"They will make $\frac{2}{5}$ of something."

"I am going to say on the blackboard '6 will make,' so that it will be ready for me to write the rest of the story when we have made the groups."

The teacher writes on the board—

6 = (As she writes the teacher should *say* it—"6 will make.")

"Did I say that the 6 objects would make all the groups?"

Removing the screen—

"What did I say they would make?" ("$\frac{2}{5}$ of something.")

"You" (to a child) "may make the groups." ()

"Is the expression finished?"

"You may finish it."

()

"What does it say?" ($\frac{2}{5}$ of 15.)

"You may write the whole story about what 6 will make." (6 = $\frac{2}{5}$ of 15.)

The teacher now completes her own story on the board—"I will finish my story now"—($6 = \frac{2}{5}$ of 15).

"You may read the story." ("6 will make $\frac{2}{5}$ of 15.")

(6)  "I have 8 objects here."

"They will make $\frac{4}{4}$ of something."

"I am going to say with the crayon '8 will make,' so that it will be ready"—

8 = (While writing the teacher gives it orally—"8 will make.")

"What did I say that 8 would make?"

"Did I say that they would make all the 4ths?"

Removing the screen—

"What did I say that they would make?"

"You may make it"— ()

"Is the expression finished?" ("Yes.")

"What does it say?" ($\frac{4}{4}$ of 8.)

"You may write the whole story about what 8 will make." ($8 = \frac{4}{4}$ of 8.)

"I will finish my whole story now"—($8 = \frac{4}{4}$ of 8).

"You may read the story." ("8 will make $\frac{4}{4}$ of 8.")

(7)  "I have 9 objects here."

"They will make $\frac{3}{7}$ of something."

The teacher writes and gives orally $9 =$ ("9 will make").

The screen is now removed.

"Will they make all the 7ths?" ("No.")

"What will they make?" ("The *seen* groups.")

"You may make the groups."

"Is the expression finished?"

"You may finish it."

"What does it say?"

"You may write the whole story about what 9 will make." ($9 = \frac{3}{7}$ of 21.)

"You may read it." ("9 will make $\frac{3}{7}$ of 21.")

(8)  "I have 12 objects here."

"They will make $\frac{3}{4}$ of something."

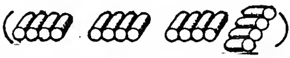
"I will begin my written story"—$12 =$ ("12 will make").

Removing the screen—

"Will they make all the groups?"

"What will they make?"

"You may make the groups and finish the expression."



"What does the expression say?" ($\frac{3}{4}$ of 16).

"You may write the whole 'will make' story." ($12 = \frac{3}{4}$ of 16.)

(9) The work should be continued with many more exercises. Expressions in $\frac{3}{3}$ of, $\frac{5}{5}$ of, $\frac{4}{4}$ of, etc., should be used occasionally. The screen should be discontinued. The question Will they make all the groups? should be asked before each construction, in order that the expressions may bring forcibly to the mind that a measurement includes *seen* groups only.

97. Measurements with Partial Choice.

Constructions similar to those in Section 96 excepting that the child has partial choice. In Section 96 the child had no choice, the teacher suggested the measurement in every case.

The teacher works at the table with large objects, but pupils at their seats also have ob-

jects and make the constructions besides making the table constructions as in Section 96.

(1)  "I have taken 8 objects."

"Let us all take 8."

"We will make the crayon say" 8=. (Orally, "8 will make.")

"These will make 2—what shall we call it?" "Shall we say 3ds, 4ths, 5ths, 6ths, 7ths, or what is your wish?"

Various pupils should be consulted as to their choice in this regard.

The teacher then settles it. "We will say 4ths."

"These 8 objects will make $\frac{2}{4}$ of something."

"Will they make all the groups?" ("No.")

"How many hidden groups must we have?"

"You" (calling some one) "may make the groups and finish the expression." ( On the teacher's desk.)

"You may make the groups on your desks and finish them."

()

"What does the expression say?" ($\frac{2}{4}$ of 16.)

"You may write the whole 'will make' story." ($8 = \frac{2}{4}$ of 16.)

(2)  "I have 14 objects."

"You may take 14."

"I will make the crayon say 14 will make"—14= (Orally, "14 will make.")

"These will make 7—shall I say 7ths, 8ths, or 9ths?" "What is your wish?"

Some will want one, some another of these.

The teacher finally chooses. She will perhaps say 9ths.

"These 14 objects will make $\frac{7}{9}$ of something."

"Will they make all the groups?" ("No.")

"How many hidden groups must we have?"

"You" (calling some one) "may make the groups and finish the expression." ()

"What does the expression say?" ($\frac{7}{9}$ of 18.)

"You may write the whole story." ($14 = \frac{7}{9}$ of 18.)

(3)  "I have 9 objects." "Let us take 9."

"I will make the crayon tell the beginning of my story"—9 = (The teacher reads as she writes it—"9 will make.")

"These will make 3—shall we say 3ds, 4ths, 5ths, 6ths, or something else?"

Pupils express their choices.

"We will try 3ds."

"These 9 objects will make $\frac{3}{3}$ of something."

"Did I say that they would make all the groups?" ("Yes.")

"Will there be any hidden groups?" ("No.")

"You" (to some child) "may make the groups on my desk and finish the expression." ()

"You may make the groups on your desks and finish them."

"What do your objects say?" ($\frac{3}{3}$ of 9.)

"You may write the whole story." ($9 = \frac{3}{3}$ of 9.)

(4)  "I have ten objects." "You may each take 10."

"You" (to some child) "may make the crayon tell the first part of our story." (10 =) (Require the child to talk as he writes—"10 will make.")

"These objects will make 5—shall we say 5ths, 6ths, 7ths, or what do you wish to say?"

"We will have them make 8ths."

"They will make $\frac{5}{8}$ of something."

"Will they make all the groups?"

"How many hidden groups must you put in?"

"You may make the groups and finish them." ()

"What do your objects say?" ($\frac{5}{8}$ of 16.)

"You may write the whole story." (10 = $\frac{5}{8}$ of 16.)

(5)  "I have 12 objects." "You may take 12."

The teacher sends some one to the blackboard to begin the story—(12 =)

"These objects make 4—what shall we say?"

Pupils suggest; teacher chooses.

"They will make $\frac{4}{7}$ of something."

"Will they make all the groups?"

"How many hidden groups must you use?"

"You may make the groups on your desks." (It must now be assumed that the full expression is to be constructed without further suggestion.)

"You may now write the whole story." ($12 = \frac{4}{7}$ of 21.)

(6) "We will all take 16."

"These objects will make 4—what do you wish to have it this time?"

"They will make $\frac{4}{5}$ of something."

"Will they make all the groups?"

"How many hidden groups must you use?"

"You may make the full expression."

"You may write the whole expression." ($16 = \frac{4}{5}$ of 20.)

(7) Several additional exercises similar to (6) should follow. In two or three of these, use should be made of such fractions as $\frac{4}{4}$, $\frac{6}{6}$, or $\frac{3}{3}$, etc.

NOTE.—Several exercises must be given in which the objects taken are made into a single group so as to "make" $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{8}$, etc., of something. This will give experience of a kind that the pupil needs but would

get only by special development work of this kind. In these exercises the child must not be permitted to use less than two hidden groups. If he suggests using *one* the teacher should *advise* that he use more. The term half must not be used in objective work. Considerable encouragement should be given to constructions of this kind (with *one* seen group).

98. Measurements with Choice Excepting as to Hidden Groups.

The aim of this step is to develop the habit of deciding beforehand the number of hidden groups to be made. The teacher dictates the number of objects to be taken in each case and the number of hidden groups to be used. The child exercises his own choice as to "seen" groups. In Section 97 the number of hidden groups was decided upon after the "seen" groups were constructed.

The teacher gives the dictations in the "will make" form—6=, 8=, etc., on the blackboard. The pupils work at their desks with objects. Each pupil has paper and pencil.

(1) 8= (Written on blackboard, the teacher expressing orally while writing—"8 will make.")

"Take 8 and make what you please, but you must have some hidden bundles with the others—*just* 2."

"Will the 8 make all the groups?"

"How many hidden groups did I tell you to put in?" ("*Two.*")

"You may make your stories."

The teacher looks at each construction to see that it is correct as to "seen" groups, and as to the *two* hidden groups.

One pupil after another is asked to read his object work. This should be done rapidly and several of the expressions written on the blackboard as read—($8 = \frac{2}{4}$ of 16, $8 = \frac{4}{6}$ of 12, $8 = \frac{1}{3}$ of 24, etc.).

Each pupil now writes his full expression on his paper.

(2) 10= (Written on blackboard and read —"10 will make.")

"Take 10 and make what you choose. We will have only *one* hidden group this time."

"Will the 10 make all the groups?"

"Will it make all the groups to be seen?"

"How many hidden groups are we to have this time?"

"You may make your stories."

The teacher examines the constructions,

pupils read their work, and then each pupil writes on his paper his full expression, as in (1).

(3) 6= (Written on blackboard and read —"6 will make.")

"How many are we to take this time?"

"We will have 3 hidden groups."

"Will 6 make all the groups?"

"Will it make all the seen groups?"

"How many hidden groups are we to have?"

"You may do the work." The teacher examines the constructions, pupils read, and each writes his story on his paper, as in (1) and (2).

(4) 12= (Written on blackboard as before.)

"How many are we to take?"

"We will not have any hidden groups this time."

"Will the 12 make all the groups?"

The constructions are made, examined, read, and written, as before.

NOTE.—Several additional exercises must be given in the same way. The construction of *one* "seen" group should be encouraged by signs of approval when pupils make them, otherwise this form of expression may not be made by the pupils. If the

pupils are found to be making *one* "seen" group too frequently, its construction should be discouraged.

Avoid constructions involving *half*. This term is too familiar to the child to make its use safe in work which is in the nature of a development.

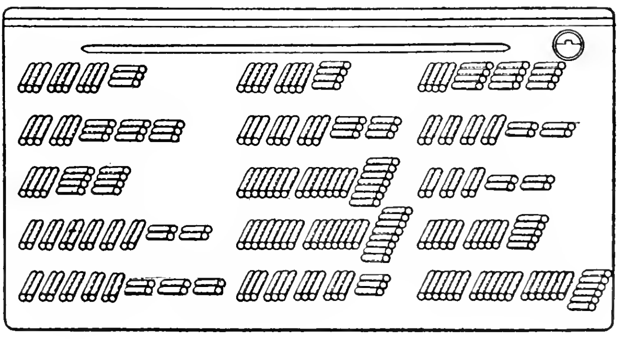
99. Seat Work in Measuring.

The pupils are now ready for seat work from blackboard dictation. The child first constructs the exercises as dictated and then writes from his constructions. This work is given day after day for several weeks, until the "memory work" is complete.

(1) Blackboard Dictation.

9 =	8 =	4 =
6 =	9 =	8 =
4 =	14 =	6 =
12 =	16 =	10 =
10 =	12 =	18 =

(2) Seat Construction Work.



(3) Pupil's Seat Written Work.

$9=\frac{3}{4}$ of 12	$8=\frac{2}{3}$ of 12	$4=\frac{1}{4}$ of 16
$6=\frac{2}{5}$ of 15	$9=\frac{3}{5}$ of 15	$8=\frac{4}{6}$ of 12
$4=\frac{1}{3}$ of 12	$14=\frac{2}{3}$ of 21	$6=\frac{3}{5}$ of 10
$12=\frac{6}{8}$ of 16	$16=\frac{2}{3}$ of 24	$10=\frac{2}{3}$ of 15
$10=\frac{5}{8}$ of 16	$12=\frac{4}{5}$ of 15	$18=\frac{3}{4}$ of 24

100. Memory Tests.

There is no oral work, no recitation, connected with the work in Section 99. When the child has made his constructions and written from them, the work is done.

After several weeks (three or four) the experiences should begin to be held in the memory. Tests should be made in the form of questions—

What will 4 make?
What will 6 make?
What will 10 make?
What will 8 make?
What else will 8 make? etc.

The questions show the form of "memory tests" to be used in this section.

If when a child is asked one of such questions, he does not reply *at once*, he should not be asked to "think" or to "try to remember." If he does not answer immediately, it indicates

that he has not had sufficient experience with that particular construction. The question should be dropped instantly, without prejudice to the child. Wait a few days and ask again. What he *knows*, however, he should have an opportunity to tell again and again.

NOTE. — For these memory tests a few minutes should be taken daily. The teacher should know what partition facts each child remembers and should question him on these facts each day if possible. Experience will add new facts to a child's stock of knowledge. These the teacher must discover by tests. In this way the range of memory work for each child will gradually enlarge. The tests for memory work for a Second Grade child should not go beyond "25 will make." Memory work in "will make" for a Third Grade child should not be urged beyond 40 or 50.

101. Memory Written Work in "Will Make."

As soon as the memory work of Section 100 is strong, seat work should be given similar to that in Section 99 but without objects.

(1) Blackboard Dictation.

8 =	10 =
4 =	14 =
10 =	6 =
12 =	9 =
9 =	4 =

(2) Pupil's Seat Written Work.

$8 = \frac{2}{3}$ of 12	$10 = \frac{2}{3}$ of 15
$4 = \frac{1}{4}$ of 16	$14 = \frac{7}{9}$ of 18
$10 = \frac{5}{6}$ of 12	$6 = \frac{1}{3}$ of 18
$12 = \frac{2}{3}$ of 18	$9 = \frac{3}{6}$ of 18
$9 = \frac{3}{5}$ of 15	$4 = \frac{2}{3}$ of 6

One division of a class may be ready for the work of Section 101 while the others are in Section 99 and Section 100. Such a division should be given the work of Section 101. It will stimulate those who are in the objective stage.

NOTE.—This Section 101 is suggested as the limit beyond which Second Grade work should not go. The partition outline may be completed with a Third Grade, but advance beyond Section 101 even with that grade should be made with caution. All the work to this point should be strong with the whole class before the sections following are taken up.

102. Oral Memory Exercises Parallel with the Work in Sections 100 and 101.

The following questions suggest the form of this work. It is recitation work that should begin after three or four weeks have been given to the seat work in Section 100.

(*a*) 9 will make what of 12? (9 will make $\frac{3}{4}$ of 12.)

4 will make what of 6? (4 will make $\frac{2}{3}$ of 6.)

5 will make what of 15? (5 will make $\frac{1}{3}$ of 15.)

8 will make what of 10? (8 will make $\frac{4}{5}$ of 10.)

(*b*) 8 is what of 12? (8 is $\frac{2}{3}$ of 12.)

10 is what of 15? (10 is $\frac{2}{3}$ of 15.)

15 is what of 20? (15 is $\frac{3}{4}$ of 20.)

After a few days another form of question should be introduced—

(*c*) 9 is what part of 12?

10 is what part of 14? etc.

NOTE.—The changes in the form of the question—from "will make" to "is" and then to "part of" are development steps. The child already has the language "will make"—4 will make $\frac{2}{3}$ of 6. By giving several questions in succession in "will make" as in (*a*), and then at once without explanation a question in "is" as in (*b*), the child immediately infers similarity of reference and answers accordingly. This develops "is." In the same way later, when the "part of" form (*c*) is suddenly used, he again infers similarity of reference, and the "part of" form is developed. These forms of question when thus approached offer no difficulty, and, while explanations would be impossible in this grade, none will be found necessary.

All three of these forms—9 will make what of 15? 9 is what of 15? and 9 is what part of 15?—should be kept in frequent use, the teacher questioning sometimes with one, at other times with another.

103. Memory Written Work in "Part Of"—to Find the Part.

This work differs from that in Section 102 only in that it is *written* work.

6 = (?) of 9.	10 = of 12.
8 = (?) of 10.	9 = of 15.
12 = (?) of 15.	12 = of 18.
12 = of 16.	12 = of 14.
10 = of 15.	10 = of 18.

The only explanation necessary in taking up this form of work will be a word stating that "6 = (?) of 8" means 6 is what of 8? or 6 is what part of 8?

This work is, of course, entirely without objects. The child writes the exercise directly from the blackboard dictation.

NOTE.—The teacher must not inquire into the pupil's mental processes in the oral work outlined above. If the child in answer to a question states that 8 is $\frac{2}{3}$ of 12, there should be no analysis, no reason, no why asked for. He has no reason. What he knows experience in measuring has taught him. The power to analyze an expression, so far from adding to his knowledge of partition, would confuse him.

104. Memory Written Work in "Part Of"—to Find the Numerator.

This is not a new step for the child. This form of dictation may be given sometimes in-

stead of the form in Section 103. Objects are not to be used. The child writes the exercise from the dictation.

Blackboard Dictation.

The following shows the form of the dictations.

$9 = {}^{?}/_{4}$ of 12.	$15 = {}^{?}/_{6}$ of 18.
$10 = {}^{?}/_{3}$ of 15.	$12 = {}^{?}/_{5}$ of 15.
$10 = {}^{?}/_{6}$ of 12.	$10 = {}^{?}/_{7}$ of 14.
$14 = {}^{?}/_{3}$ of 21.	$8 = {}^{?}/_{5}$ of 10.
$15 = {}^{?}/_{4}$ of 20.	$8 = {}^{?}/_{7}$ of 14.

The only preliminary exercise to this is a word to explain that "$8 = {}^{?}/_{6}$ of 12?" says 8 is how many 6ths of 12? Exercises in this form and in the form given in Section 105 below should not be as frequent as those outlined in Section 103.

105. Memory Written Work in "Part Of"—to Find the Denominator.

This form of work may sometimes be given instead of that in Section 103.

Objects are not to be used.

Blackboard Dictation.

The following shows the form of the blackboard dictations for the work of this section:

$10 = {}^{5}/_{?}$ of 14.	$18 = {}^{3}/_{?}$ of 24.
$12 = {}^{4}/_{?}$ of 15.	$12 = {}^{3}/_{?}$ of 16.
$4 = {}^{1}/_{?}$ of 12.	$14 = {}^{2}/_{?}$ of 21.
$6 = {}^{3}/_{?}$ of 10.	$5 = {}^{1}/_{?}$ of 20.
$7 = {}^{1}/_{?}$ of 21.	$9 = {}^{3}/_{?}$ of 15.

NOTE.—It must be understood that while the written work in Sections 103, 104, and 105 is being done there are daily oral exercises like those in Section 102. Oral work like the written work in Sections 104 and 105 may also be given.

106. Partition Reversed.

We have so far dealt with partition in the form $9 = \frac{3}{4}$ of 12—partition in "will make" form. It should now be developed in the form in which it occurs in common experiences.

(1) Oral Work in Turning the Expression Around.

This work may take form as follows:

(*a*) "I have behind this book $\frac{2}{3}$ of 9."

"What did I make it from?" The answer will come at once—("You made it from 6").

NOTE.—It is not necessary that there should be anything behind the book. The child will not be deceived. He may know that there is nothing there, but the screen will stimulate imagination and attention. It makes for vividness.

"Yes, $\frac{2}{3}$ of $9=6$." (The teacher writes it on the blackboard, "talking" as she writes, the words "I made it from" being used as the crayon forms the sign $=$.) The full "talk" with the writing is "two-thirds of nine, I made it from six."

(*b*) "I have behind this book $\frac{4}{5}$ of 10."

"What did I make it from?" ("You made it from 8.")

"Yes" (the teacher talking as she writes on the blackboard), $\frac{4}{5}$ of $10=$ ("I made it from") 8.

(*c*) "I have here $\frac{2}{3}$ of 12."

"What did I make it from?" ("You made it from 8.")

"Yes" (writing on the blackboard and talking), $\frac{2}{3}$ of $12=8$.

(2) Written Work in Turning the Expression Around.

The teacher now writes the expression on the blackboard as she announces it—"talking" as she makes the figures:

(*a*) "I have $\frac{5}{6}$ of 12." "What did I make it from?" ("You made it from 10.")

"Yes," $\frac{5}{6}$ of $12=10$. The teacher does not rewrite the first part of the expression on the blackboard; she

simply completes the expression by adding the 10, using the expression "I made it from 10" as she writes the " = 10."

(*b*) $\frac{3}{4}$ of 12 =

"What did I make this from?" ("You made it from 9.")

"You" (indicating some child) "may complete the expression on the blackboard."

(*c*) $\frac{4}{5}$ of 15 =

"What did I make this from?" ("You made it from 12.")

"You may finish the expression."

NOTE.—Two or three more such exercises will make the pupils familiar with the reversed form so that it can be used for seat work.

107. Memory Written Work in Partition Reversed.

The following shows the form of the blackboard dictation work. This work is of course without objects.

Blackboard Dictation.

$\frac{3}{4}$ of 12 =	$\frac{6}{7}$ of 14 =
$\frac{2}{3}$ of 9 =	$\frac{7}{8}$ of 8 =
$\frac{4}{5}$ of 10 =	$\frac{1}{4}$ of 16 =
$\frac{5}{6}$ of 12 =	$\frac{2}{3}$ of 15 =
$\frac{3}{4}$ of 16 =	$\frac{5}{7}$ of 14 =

NOTE.—The introduction of this *reversed* form of partition must not be understood to do away with work like that in Sections 103, 104, and 105. Seat work in each of these four forms, especially in the forms outlined in Sections 103 and 107, must be continued until all partition work has been completed.

108. Oral Work on Section 107.

After a few days, oral tests on the work of reversed partition should begin. The questions in these tests may be in the following forms in order:

(*a*) "$\frac{3}{4}$ of 12." "What do we make it from?"
(*b*) "$\frac{3}{4}$ of 12." "How many in $\frac{3}{4}$ of 12?"
(*c*) "$\frac{3}{4}$ of 12." "How many?"
(*d*) "$\frac{3}{4}$ of 12 are how many?"

NOTE. — These questions indicate development steps. Give a few exercises in the form given in (*a*), then put several into form (*b*), at the proper time give one in form (*c*), later use form (*d*). It may be the second or third lesson period before (*c*) or (*d*) could be safely given. Very much of this oral work should be given.

109. General Oral Work in Partition.

As soon as the oral work in Sections 102 103, 104, 105, 107, and 108 is strong and rapid, oral exercises like the following should begin and should form a part of the general oral drills in partition.

$\frac{3}{4}$ of 8 is what part of 18?
$\frac{2}{3}$ of 12 is what part of 10?
$\frac{5}{6}$ of 12 is what part of 15?

The introduction of this work requires no preliminary steps. In giving the first question of this kind the teacher should simply make a long pause after the first part of the question, then give the rest of the question. For instance, $\frac{3}{4}$ of 8—is what part of 18? The question is not difficult as soon as the child understands what is meant. $\frac{2}{3}$ of 12—is what part of 10? As pupils begin to grasp the meaning, the dictations may be given with shorter and shorter pauses. In a few days expressions of that kind may be dictated without a pause.

Exercises of this kind are very valuable for developing rapid thinking. They are of value also in the development of partition imagery.

NOTE.—This completes proper-fraction partition.

IMPROPER-FRACTION PARTITION.

110. The Meaning of the Term.

In our work thus far we have used only partition expressions of wholes or less than wholes —$\frac{4}{4}$ of 12, $\frac{3}{5}$ of 15, $\frac{2}{3}$ of 9, $\frac{3}{3}$ of 18, etc. We must now give attention to expressions of more than wholes—$\frac{4}{3}$ of 9, $\frac{8}{6}$ of 12, $\frac{6}{4}$ of 8, etc.

Expressions of this kind we have arbitrarily called improper-fraction expressions. $\frac{4}{4}$ of 8, $\frac{3}{3}$ of 12, etc., we have classed with the proper-fraction expressions, because they express wholes.

With pupils who have had the full proper-fraction partition work, this branch of partition requires little more than an explanation of one or two expressions and instruction in the form of the concrete expression.

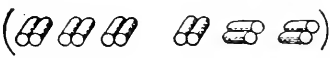
111. The Language of Improper-fraction Partition.

The pupils are at their desks with objects; the teacher at her desk with large objects.

(1) "Show me $\frac{4}{3}$ of 6." (The teacher writes $\frac{4}{3}$ of 6 on the blackboard.)

Pupils will indicate by look and action that they think the expression impossible. This is natural, because "all the 3ds of 6" have so far been $\frac{3}{3}$ of 6.

The teacher explains—"this means all the 3ds of 6 and *one* of the 3ds of another 6. I will make it for you"—



NOTE.—It must be noted that a wide space is left between the thirds of the two 6's. This shows to the eye "all the thirds of 6 and one of the thirds of the

other 6." In all construction work in improper-fraction partition there must be this wide space between the parts of the wholes, otherwise the child will lose sight of the full meaning of the expressions. There will be nothing in the child's thought in this work that will not show in his constructions. On the other hand, the constructions will indicate that he has had this thought. $\frac{7}{3}$ of 6 when constructed would be 

The construction shows the three wholes.

"You may make $\frac{4}{3}$ of 6."

(2) "Show me $\frac{6}{4}$ of 8" (written on the blackboard—$\frac{6}{4}$ of 8).

"This means all the 4ths of 8 and 2 of the 4ths of another 8."

"Let us all make it."

(3) "Show me $\frac{5}{3}$ of 9" (written, $\frac{5}{3}$ of 9).

"It means all the 3ds of 9 and 2 of the 3ds of another 9."

"We will all make it."

(4) "Show me $\frac{7}{4}$ of 12."

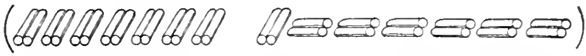
"You may tell me what it means." ("It means all the 4ths of 12 and three of the 4ths of another 12.")

"You may make it."

(5) "Show me $\frac{8}{7}$ of 14."

"You may tell me what it means."

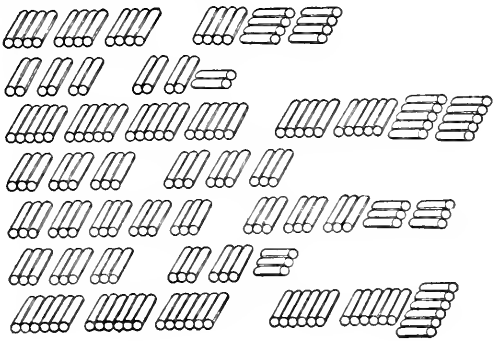
"You may make it."



(6) Several more exercises should be given, the pupils in each exercise being asked (1) to tell what it means and (2) to construct.

112. Reading Lessons in Improper-fraction Partition.

The teacher should prepare two or more graphic objective reading lessons like the following for pupils to interpret orally. This is the necessary preparation for reading seat construction work later.

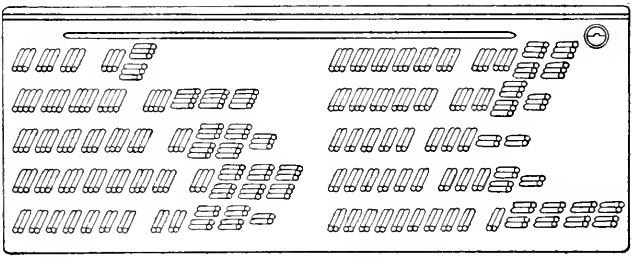


113. Seat Work in Improper-fraction Language.

(1) Blackboard Dictation.

$\frac{4}{3}$ of 9	$\frac{8}{6}$ of 18
$\frac{5}{4}$ of 16	$\frac{7}{5}$ of 15
$\frac{7}{6}$ of 18	$\frac{8}{5}$ of 10
$\frac{8}{7}$ of 21	$\frac{9}{6}$ of 12
$\frac{9}{7}$ of 14	$\frac{10}{9}$ of 18

(2) Pupil's Seat Work—Objective.



(3) After the construction has been finished, the dictation is erased and the pupils write the expressions from the object work on their desks.

114. Improper-fraction Partition Measuring.

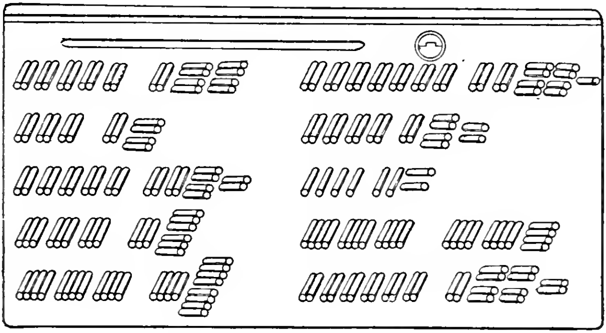
It will not be necessary to make any development of "will make" work in this department of partition. When the child has the language as in Section 113, he is ready to use it at once in measuring. He has already had

much experience in partition "will make" work and will readily pass to this new phase.

(1) Blackboard Dictation.

12 =	18 =
8 =	10 =
14 =	6 =
12 =	20 =
16 =	14 =

(2) Construction Work. What some child may construct.



(3) After the construction work is done the pupils write the exercises from their desk work.

NOTE.—The work of Section 114 must be continued several weeks. The suggestions under Section 100 about memory tests and those in Sections 101 and 102 for written and oral memory work are applicable here.

All the forms of memory work suggested for proper-fraction partition must be kept up throughout the whole partition period and for months afterwards.

After the memory work in improper-fraction partition is completed, the oral partition exercises should consist of questions involving now one and now the other of these two partition forms.

The partition work here outlined may easily be completed in the Third Grade. Rapid oral exercises, however, should be given frequently throughout the next two grades.

This completes the partition work and the development of the five fundamental counting (measuring) processes of number.